Number Patterns / Sequences

4.1 Generating arithmetic expressions from repeating Patterns and the rules that govern them; students construct an understanding of a relationship as that which involves a set of inputs, a set of outputs and a correspondence from each input to each output.

Number of units V Cost

2014 Question paper 1.2012 Question 7 2012 Sample Question 9 Paper 1 Paper 1

Use tables to represent a repeating-pattern situation – generalise and explain patterns and relationships in words and numbers – write arithmetic expressions for particular terms of a sequence.

This enables us to use the formula from the tables for the $T_n = a + (n-1)d$

of an arithmetic sequence

4.2 Representing situations with tables, diagrams and graphs Relations derived from some kind of context – familiar, everyday situations, imaginary contexts or arrangements of tiles or blocks (2015

Students look at various patterns and make predictions about what comes next. (2014)

Use tables, diagrams and graphs as tools for representing and analysing linear, quadratic and exponential patterns and relations (exponential relations limited to doubling and tripling) – develop and use their own generalising strategies and ideas and consider those of others – present and interpret solutions, explaining and justifying methods, inferences and reasoning

4.3 Finding formulae Ways to express a general relationship arising from a pattern or context. – **find the underlying formula written in words from which the data is derived (linear relations) –**

Find the underlying formula algebraically from which the data is derived (linear, quadratic relations)

We need to be able to find T_n for a linear (Arithmetic sequence) $T_n = a + (n-1)d$ and T_n of a quadratic sequence $T_n = an^2 + bn + c$ where

$2a = (d_2)$ $3a + b = T_2 - T_1$ **and** $a + b + c = T_1$

4.4 Examining algebraic relationships Features of a relationship and how these features appear in the different representations.

Constant rate of change: linear relationships ($y = mx$.)

Non-constant rate of change: quadratic relationships $T_n = an^2 + bn + c$.

Proportional relationships (y = mx) a line through (0,0) with slope m . Non proportional relationships (y = mx + c) a line through (0,c) with slope m .Show that relations have features that can be represented in a variety of ways – distinguish those features that are especially useful to identify and point out how those features appear in different representations: **in tables, graphs, physical models, and formulas expressed in words, and algebraically** – use the representations to reason about the situation from which the relationship is derived and communicate their thinking to others – **recognise that a distinguishing feature of quadratic relations is the way change varies, (2015 Sample Question 8 paper 1) (2014 Question 8 paper 1) 2013 Question 8 paper 1**.

Discuss rate of change and the y-intercept, **(2014 Question 9 paper 1**)consider how these relate to the context from which the relationship is derived, and identify how they can appear in a table, in a graph and in a formula –

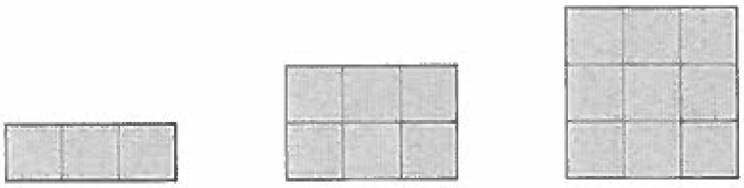
Decide if two linear relations have a common value (decide if two lines intersect and where the intersection occurs) **Simultaneous equations**

Investigate relations of the form y=mx and y=mx +c – recognise problems involving direct proportion and identify the necessary information to solve them

Number Patterns Sequences

Case 1: Given a number pattern .
Method :(i)Write out the pattern
(ii)Find the first differences if $T_2 - T_1 = T_3 - T_2$ if the first differences are a constant (d_1) then the Sequence is Arithmetic (linear).(iii) The general term can be expressed in the form $T_n = a + (n-1)d$ where $a = T_1$ and $T_2 - T_1 = d$.

Example 1



Pattern 1　　Pattern 2　　Pattern 3

Term	T_1	T_2	T_3	T_4
Pattern	3	5	7	9
d_1	2	2	2	

$T_2 - T_1 = T_3 - T_2 = 2$ the sequence is arithmetic (linear)

We can us $T_n = a + (n-1)d$ to find the general term $T_n = 3 + (n-1)2 = 3 + 2n - 2 = 2n + 1$

Example 2 Study the number of squares in the diagram below and investigate if the pattern is linear ,quadratic or exponential.

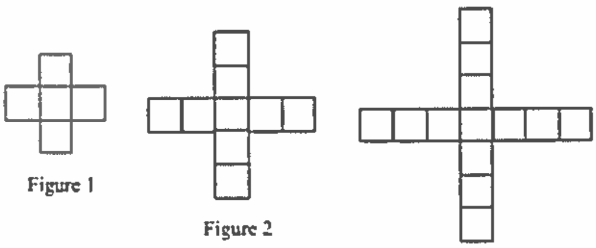


Figure 1　　Figure 2　　Figure 3

Find the general term T_n

Term	T_1	T_2	T_3
Pattern	5	9	13
d_1	4	4	

The pattern is linear $T_2 - T_1 = T_3 - T_2 = 4, T_n = a + (n-1)d$

$T_n = 5 + (n-1)4 = 5 + 4n - 4 = 4n + 1$

Proportional and non proportional relationships

To find the general term for a number pattern based on a table

We can be given two types of tables

Example 3: (i)Type 1 ; This is where x is directly Proportional to y

X Y

Hours	Money
0	$0
1	$9
2	$18
3	$27
4	$36

Solution:

The Expression for the rule can be found by using the formula $y - y_1 = m(x - x_1)$

The slope is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-0}{1-0} = 9 = m$, $y - y_1 = m(x - x_1) \rightarrow y - 0 = 9(x-0) = y = 9x$

Interpretation of the slope is for every one hour increase in time there is an increase of $9 in money. This is a line through (0,0) slope 9.

Example 4:

x	0	2	4	6
y	4	10	16	22

The Expression for the rule can be found by using the formula

$y - y_1 = m(x - x_1)$ $\frac{y_2 - y_1}{x_2 - x_1} = \frac{10-4}{2-0} = 3 = m$

$y - y_1 = m(x - x_1) \rightarrow y - 4 = 3(x-0) = y = 3x + 4$

Or using $y = mx + c, m = 3 \rightarrow y = 3x + c$ sub in (0,4) $4 = m(0) + c \rightarrow c = 4$

This is a non proportional relationship.

This is a line with slope 3 and an intercept with the y axis (0,4)

This type of problem was asked in 2012 (Electricity + standing charge)

Quadratic Sequences

$$T_n = an^2 + bn + c$$

$T_1 = a + b + c$,

$T_2 = a(2)^2 + b(2) + c = T_2 = 4a + 2b + c$,

$T_3 = a(3)^2 + 3b + c \rightarrow T_3 = 9a + 3b + c$

If we find the differences between the terms $T_2 - T_1 = 3a + b = d_1$ $T_3 - T_2 = 5a + b = d_1$

Now find the 2nd difference $5a + b - (3a + b) = 2a$.

For all quadratic sequences the 2nd differences are all 2a.

We can use this fact if we know T_1, T_2, T_3, **to find a,b,c**

Give the sequence 2,9,22,41,66

Prove the sequence is Quadratic and find general term .

First differences are 7,13,19,25.

The 2nd differences are 6,6,6,6 since the 2nd differences are all 6 the sequence is quadratic

Solution:

We know (i) $2a = 6 \rightarrow a = 3$.

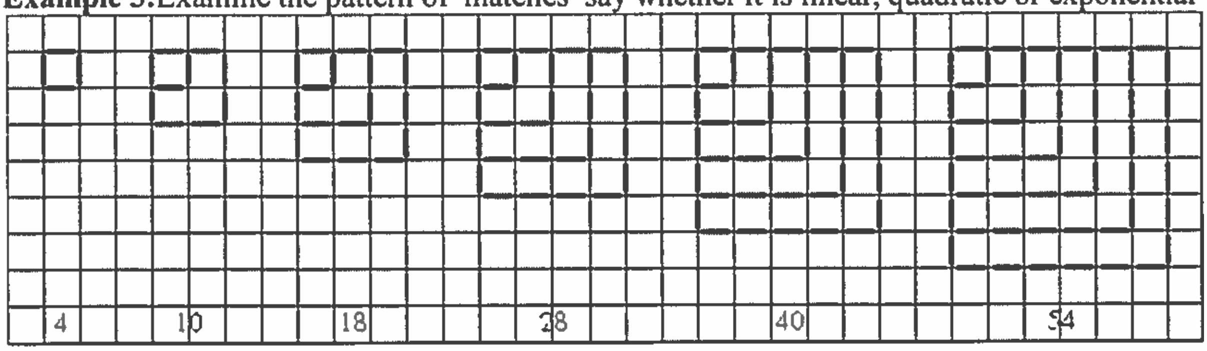
We know (ii) $3a + b = 7(T_2 - T_1) \rightarrow 3(3) + b = 7 \rightarrow b = -2$,

We know(iii) $T_1 = a + b + c \rightarrow 3 - 2 + c = 2 \rightarrow c = 1$

$T_n = 3n^2 - 2n + 1$

Quadratic Sequences

Example 5: Examine the pattern of matches say whether it is linear, quadratic or exponential



Solution

Term	T_1	T_2	T_3	T_4	T_6	T_6
Pattern	4	10	18	28	40	54
d_1	6	8	10	12	14	
d_2	2	2	2	2		

(i) Since the 2nd differences are equal then the pattern is Quadratic

(ii)Find an expression for the general term T_n .

$T_n = an^2 + bn + c$ .We know $2a = (d_2) \rightarrow a = 1$, **and** $3a + b = T_2 - T_1 \rightarrow 3(1) + b = 6 \rightarrow b = 3$

We also know $a + b + c = T_1 \rightarrow 1 + 3 + c = 4 \rightarrow c = 0$ $T_n = an^2 + bn + c = n^2 + 3n$

How many matches are in the 10th pattern?

$T_{10} = n^2 + 3n = (10)^2 + 3(10) = 130$

Example 6: Examine the pattern below and state whether it is Linear, Quadratic or Exponential

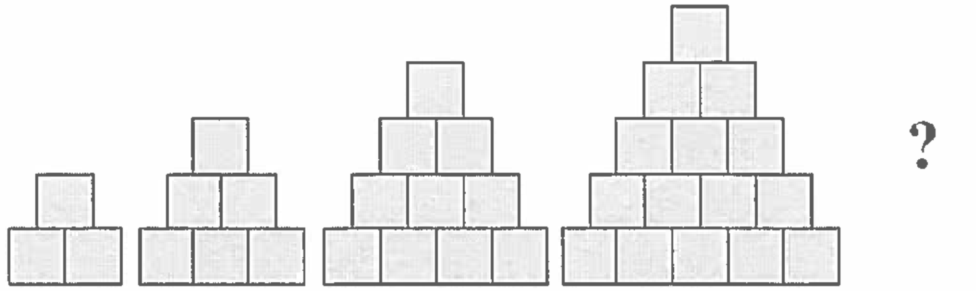


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

Term	T_1	T_2	T_3	T_4	T_5
Pattern	3	6	10	15	**21**
d_1	3	4	5	6	
d_2	1	1	1		

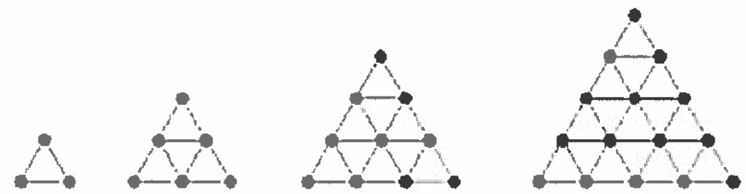
(i) Since the 2nd differences are equal then the pattern is Quadratic (ii) $T_n = an^2 + bn + c$ ..

$2a = (d_2) \to a = \frac{1}{2}$, and $3a + b = T_2 - T_1 \to 3(\frac{1}{2}) + b = 3 \to b = \frac{3}{2}$.

$.5 + 1.5 + c = T_1 \to .5 + 1.5 + c = 3 \to c = 1$ $T_n = an^2 + bn + c \Longrightarrow .5n^2 + 1.5n + 1$

$T_5 = 0.5(5)^2 + (1.5)5 + 1 = 21$

This is the problem asked on the 2014 exam



Pattern	T_1	T_2	T_3	T_4	T_n
Number of Dots Pattern 1	3	6	10	15	
d_1	3	4	5		
d_2	1	1		**Quadratic**	$T_n = 0.5n^2 + 1.5n + 1$
Number of triangles Pattern 2	1	4	9	16	
d_1	3	5	7		
d_2	2	2		**Quadratic**	$T_n = n^2$
Number of matches Pattern 3	3	9	18	30	

d_1	6	9	12		
d_2	3	3		**Quadratic**	$T_n = 1.5n^2 + 1.5n$

To Calculate: $T_n = an^2 + bn + c$ for the above quadratic sequences.
We use the following facts
(i) $d_2 = 2a$,(ii) $3a + b = T_2 - T_1$ (iii) $a + b + c = T_1$

Pattern 1(Dots)
(i) $d_2 = 2a = 1 \rightarrow a = 0.5$ (ii) $3a + b = T_2 - T_1 \rightarrow 3(.5) + b = 3 \rightarrow b = 1.5$ (iii)
$a + b + c = T_1 \rightarrow 0.5 + 1.5 + c = 3 \rightarrow c = 1$ $T_n = 0.5n^2 + 1.5n + 1$

Pattern 2: (Triangles) (i) $d_2 = 2a = 2 \rightarrow a = 1$ (ii) $3a + b = T_2 - T_1 \rightarrow 3(1) + b = 3 \rightarrow b = 0$
$a + b + c = T_1 \rightarrow 1 + 0 + c = \Rightarrow c = 0$ $T_n = n^2$

Pattern 3;Number of matches ;
(i) $d_2 = 2a \rightarrow 2a = 3 \rightarrow a = 1.5$ (ii) $3a + b = T_2 - T_1 \rightarrow 3(1.5) + b = 6 \rightarrow b = 1.5$
(iii) $a + b + c = T_1 \rightarrow 1.5 + 1.5 + c = 3 \rightarrow c = 0$
$T_n = 1.5n^2 + 1.5n$

Squares pattern

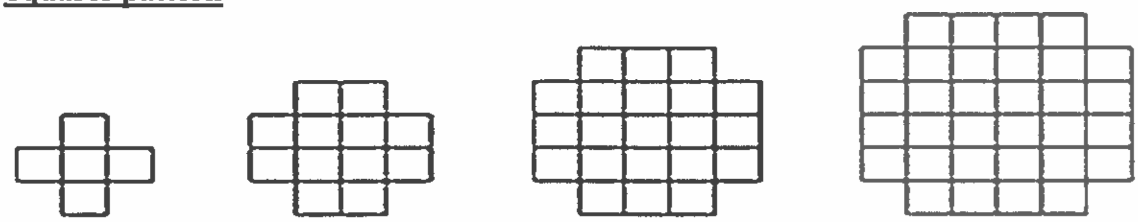


Figure 1 Figure 2 Figure 3 Figure 4

TILE PATTERN

Figure	1	2	3	4
Perimeter	12	16	20	24

d_1	8	8	8

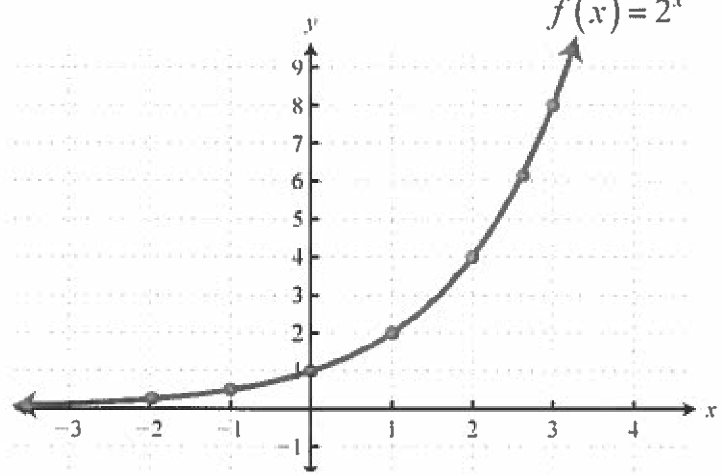
The pattern above is arithmetic $T_2 - T_1 = T_3 - T_2$

Figure	1	2	3	4
Area (number of squares)	5	12	21	32
d_1	7	9	11	
d_2	2	2		

The pattern above is quadratic

Exponential Patterns (Geometric sequences)

T_n	T_1	T_2	T_3	T_4
2^n	2	4	8	16
$\frac{t_2}{t_1}=\frac{t_3}{t_2}$				$\frac{4}{2}=\frac{8}{4}=2$
3^n	3	9	27	81
$\frac{t_2}{t_1}=\frac{t_3}{t_2}$				$\frac{9}{3}=\frac{27}{9}=3$



Time seconds	1	0.5	1	1.5
Height above the ground	12.4	17.1	16.6	10.9

The table (left) shows the height above the ground at a given time t.

Show the sequence of heights is quadratic
(i)Find the general term and find the time when the object is back on the ground.

Term	1	2	3	4
	12.4	17.1	16.6	10.9
d_1	4.7	-0.5	-5.7	
d_2	-5.2	-5,2		

The sequence is quadratic
$d_2 = 2a = -5.2 \to a = -2.6$ $3a + b = T_2 - T_1 \to 3(-2.6) + b = 4.7 \to b = 12.5$

$a + b + c = T_1 \to -2.6 + 12.5 + c = 12.4 \to c = 2.5$
$T_n = -2.6n^2 + 12.5n + 2.5$
The object is on the ground when $T_n = -2.6n^2 + 12.5n + 2.5 = 0$

Use $n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-12.5 \pm \sqrt{(12.5)^2 - 4 \times (-2.6)(2.5)}}{2(-2.6)} = n = 5, n = -\frac{5}{26}$

The 5th term of the sequence represents 2 seconds.