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Probability and Statistics

Financial Maths

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Probability and Set Notation
Notes for Project Maths students

Set Notation
$A \cup B$: A union B this is the set of elements which belong to both sets A and B with none repeated.
$A \cap B$: A intersection B this is the set of all the elements that A and B have in common.
Independent events.
Two or more events are said to be independent if the events have no influence on each other.
Probability and set notation.
If A and B are independent events
Rule 1 $P(A \cap B) = P(A).P(B)$
Rule 2 $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Conditional probability
Rule 3: $P(A / B)$ = the conditional probability that event A occurs given that event B has already occurred.
Rule 4: $P(B / A)$ =the conditional probability that event B occurs given that event A has already occurred.
$P(A \cap B) = P(A / B)P(B)$ or $P(A \cap B) = P(B / A)P(A)$.

Sample Questions

Question 1 Trial <u>Project maths</u> paper

The events A and B are such that P (A) = 0 .7 P (B) = 0.5 and $P(A \cap B) = 0.3$
Find (i) $P(A \cup B)$
Solution $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $P(A \cup B) = 0.7 + 0.5 - 0.3 = 0.9$
<u>(5 marks)</u>
Find (ii) $P(A / B)$
From above we know

$$P(A \cap B) = P(A / B)P(B) \Rightarrow P(A / B) = \frac{P(A \cap B)}{P(B)} = \frac{0.3}{0.5} = 0.6$$ **(10 marks)**

State whether A and B are independent events.
If A and B are independent then $P(A \cap B) = P(A).P(B)$ check $0..3 \neq (0.7)x(0.5) = 0.35$
(10 Marks)

Question 2 <u>Project Maths Mock</u> 2010

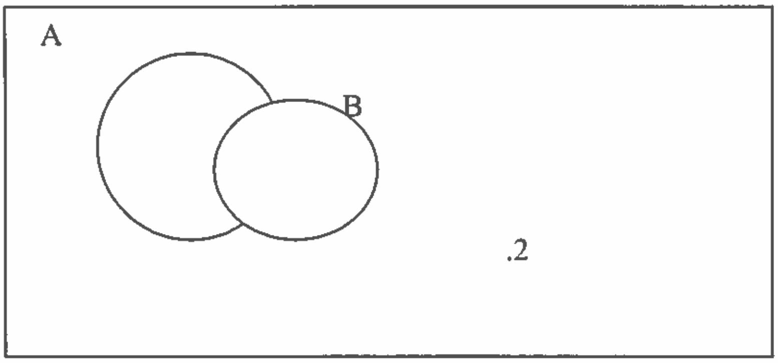
Given
P (A) = $\frac{1}{2}$, P(B)= $\frac{1}{3}$ and $P(A \cup B) = \frac{3}{4}$
(i)Find $P(A \cap B)$ We know that
$P(A \cup B) = P(A) + P(B) - P(A \cap B) \Rightarrow P(A \cap B) = P(A) + P(B) - P(A \cup B) \Rightarrow$
$P(A \cap B) = \frac{1}{2} + \frac{1}{3} - \frac{3}{4} = \frac{1}{12}$
State whether A and B are independent;

If A and B are independent $P(A \cap B) = P(A).P(B)$ check $\frac{1}{12} \neq \frac{1}{2} \times \frac{1}{3}$ A and B are not independent.

2010 Leaving cert Higher Project Maths Q1 Section A.

Two events A and B are such that P(A) = .2 , $P(A \cap B) = .15$ and $P(A' \cap B) = .6$

(a)Complete this Venn diagram



Note the sum of the probabilities = 1 .

Given P(A) = .2 and $P(A \cap B) = .15$ $\Rightarrow P(A..only) = 0.05$ and $P(A' \cap B) = .6$ which is the same as P(B only).

(b)Find the probability neither A or B happens.

What they are looking for $P(A \cup B)'$ (everything outside A and B)

Ans .2

© Find the conditional probability $P(A \setminus B)$.

We use the rule $P(A \cap B) = P(A/B)P(B) \Rightarrow \frac{P(A \cap B)}{P(B)} = P(A \setminus B) = \frac{.15}{.75} = .2$

(d)State whether A and B are independent events and justify your answer .

A and B are independent events if P(A).P(B)= $P(A \cap B)$ =(.02)x(.75)=0.15 True.

2009 Leaving Cert Higher Paper 2 Further Probability option.

A and B are independent events such that P (A) = .25 and $P(A \cup B) = .55$ Find P (B)

Since A and B are independent

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A).P(B)$

$\Rightarrow 0.55 = 0.25 + P(B) - 0.25P(B) \Rightarrow 0.75P(B) = 0.3 \Rightarrow P(B) = 0.4$

2007 Leaving Cert Higher Maths paper 2 further probability option.

Two events A and B are independent such that P (A) = $\frac{1}{5}$ and P (B) = $\frac{1}{7}$ find $P(A \cap B)$

Since A and B are independent $P(A \cap B) = P(A).P(B) \Rightarrow P \cap B = \frac{1}{5} \times \frac{1}{7} = \frac{1}{35}$

(ii) Find $P(A \cup B)$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{5} + \frac{1}{7} - \frac{1}{35} = \frac{11}{35}$

2006 Leaving Cert Higher Maths Paper 2 further probability option.
A bag contains the following cardboard shapes.
10 red squares, 15 green squares, 8 red triangles and 12 green triangles
One of the shapes is drawn at random from the bag.

A is the event that a square is drawn, $\rightarrow P(A) = \dfrac{25}{45}$

B is the event that a green shape is drawn. $P(B) = \dfrac{27}{45}$

(i)Find $P(A \cap B)$ this is the probability that a green square is drawn $\dfrac{15}{45}$

(iii)Find $P(A \cup B)$

We know $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{25}{45} + \dfrac{27}{45} - \dfrac{15}{45} = \dfrac{37}{45}$

(i)State whether A and B are independent events. If A and B are independent then

$P(A \cap B) = P(A).P(B) \rightarrow \dfrac{15}{45} = \dfrac{1}{3} = \dfrac{25}{45} \times \dfrac{27}{45} = \dfrac{1}{3}$ true then A and B are independent

(ii)State whether if A and B are mutually exclusive events

A and B are mutually exclusive if $A \cap B = \phi$ this is not true as $(A \cap B) = \dfrac{15}{45}$ or

$P(A \cup B) = P(A) + P(B) \rightarrow \dfrac{37}{45} \neq \dfrac{25}{45} + \dfrac{27}{45}$.

2002 Leaving Cert Higher maths paper 2 further probability.
P (E\F) denotes the conditional probability of E given F.
Write down an equation to express the relationship between P (B), P (A|B), $P(A \cap B)$
We know from above that $P(A \cap B) = P(A \setminus B)P(B)$
Given A and B events such that

$P(A \setminus B) = \dfrac{1}{2}, P(B \setminus A) = \dfrac{1}{3}, P(A \cap B) = \dfrac{1}{7}$

We are asked to find $P(A \cup B)$.
We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ we need to find P (A) and P (B) first

$P(A) = \dfrac{P(A \cap B)}{P(B \setminus A)} = \dfrac{1/7}{1/3} = \dfrac{3}{7} : P(B) = \dfrac{P(A \cap B)}{P(A \setminus B)} = \dfrac{1/7}{1/2} = \dfrac{2}{7}$

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \dfrac{3}{7} + \dfrac{2}{7} - \dfrac{1}{7} = \dfrac{4}{7}$

Are A and B are independent events?

$P(A \cap B) = P(A).P(B) \Rightarrow P \cap B = \dfrac{1}{7}, P(A).P(B) = \dfrac{2}{7} \times \dfrac{4}{7} = \dfrac{8}{49} \neq \dfrac{1}{7}$

Therefore the events are not independent.

1998 leaving cert higher maths paper 2 further probability.

Given $P(A)=\frac{1}{2}, P(B)=\frac{2}{3}, P(A\cap B)=\frac{1}{3}$ find $P(A\cup B)$

We know that $P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{2}+\frac{2}{3}-\frac{1}{3}=\frac{5}{6}$

If A and B are two mutually exclusive events and
$P(A)=.1, P(B)=.3,$ find $P(A\cup B)$.
$P(A\cup B)=P(A)+P(B)=P(A)+P(B)=.1+.3=.4$
If A and B are independent events and $P(A)=.3, P(B)=.6,$
Find P(A and B) = $P(A\cap B)=P(A).P(B)\rightarrow .3\times .6=.18$

Find P($\bar{A}$ and B) = P(1-A) X P(B) .P($\bar{A}$)= 1 –P(A)=1-0.3 = 0.7

Therefore P($\bar{A}$ and B)= .7 x .6 = .42

Main Points

Probability is measured on a scale from 0 to 1 .
Two events are **mutually exclusive** if they cannot both happen at the same time.
If A and B are mutually exclusive events then $P(A\cap B)=0$
The event A not happening as a result of a trial is called the compliment of A written A'
P (A') = $1-P(A)$
Two events are **independent** if the probability of one happening is unaffected by whether or not the other happens. A and B are independent events if $P(A\cap B)=P(A).P(B)$
P(A|B) denotes the probability that event A happens given that event B has happened already
If A and B are independent P(A|B)=P(A) and $P(A\cap B)=P(A).P(B\mid A)=P(B).P(A\mid B)$

Probability and the Binomial theorem

This area of the course involves repeated trials of the same experiment each trial is independent eg tossing a coin 4 times or throwing a dice 6 times or taking 9 penalties or answering your mobile when it rings on 5 occasions . There are only two solutions to the trial, **success or failure.** If p is the probability of a success and q is the probability of failure when n trials take place the probability of **r** successes in **n** trials is

$P(r)={}^{n}_{r}C(p)^{r}(q)^{n-r}$ this is really just the general term of the binomial expansion of $(p+q)^{n}$
Note $(p+q)=1$

Examples

Project Maths Pre-Leaving cert Exam Feb 2010 Question 5.n (25 marks)
The following formula relates to the binomial distribution.
$P(X=r)={}^{n}_{r}C(p)^{r}(q)^{n-r}$
(i)**State what each letter p,q,n and r represent** .
Answers
(i)p is the probability of success (ii) q is the probability of failure (iii) n is the number of trials (iv) r is the number of successes .

(ii) **Describe the type of experiment that results in a random variable that has a binomial distribution.**

Answer ;

Repeated trials that are independent of each other in which there are only two possible outcomes success or failure .

Example: Tossing a coin **n times** and looking for the probability of getting **r heads.**

Question

(b)In an archery competition Laura hits the target with an average of two out of every three shots.During one competition she has 10 shots at the target.

(i)Find the probability that Laura hits the target nine times .

Solution :

N = 10 ,r = 9,p = $\frac{2}{3}, q = \frac{1}{3}$ the probability that She it the **hits the target 9 times** is

$P(r) = {}^{10}_{9}C(\frac{2}{3})^9(\frac{1}{3})^1 = 0.0867.$

(ii)Find the probability Laura hits the target less than 9 time .

There are 2 ways to do this (i)Find the probability Laura hits the target once or twice etc .

Or 1 – (P(9) + P(10))

P(9) = 0.0867 , $P(r) = {}^{10}_{10}C(\frac{2}{3})^{10}(\frac{1}{3})^0 = 0.0173$

$\Rightarrow P(<9hits) = 1-(0.0867+0.0173) = 0.8960$

2010 Leaving cert Higher Maths Question 9 further probability .

A test consists of 20 multiple choice questions .Each question has 4 possible answers ,only one of which is correct. Sean decides to guess all the answers at random.

N = 20 p = ¼, q = ¾

Find the probability that

(i)Sean gets no answers correct Ans $^{20}\ C(\frac{1}{4})^0(\frac{3}{4})^{20} = 0.003$

$c0$

(ii)Sean gets exactly five answers correct. Ans ${}^{20}_{5}C(\frac{1}{4})^5(\frac{3}{4})^{15} = 0.202$

(iii)Sean gets 4,5,or 6 answers correct ${}^{20}_{4}C(\frac{1}{4})^4(\frac{3}{4})^{16} + {}^{20}_{5}C(\frac{1}{4})^5(\frac{3}{4})^{15} + {}^{20}_{6}C(\frac{1}{4})^6(\frac{3}{4})^{14} = 0.561$

Give your answer correct to 3 places of decimals.

2008 Leaving Cert Higher maths further probability Question 9 paper 2 .

20% of the items produced by a machine are defective,4 items are chosen at random .Find the probability none of the chosen items are defective .

Solution $P(r) = {}^{n}_{r}C(p)^{r}(q)^{n-r}$: $n = 4, p = \frac{1}{5}, q = \frac{4}{5}, r = 0,1,2,3,4$ The probability that none are defective is r = 0, $P(r) = {}^{4}_{0}C(\frac{1}{5})^{0}(\frac{4}{5})^{4} = \frac{256}{625}$ or ND .ND.ND.ND = $\left(\frac{4}{5}\right)^{4}$

2007 Leaving Cert Higher maths further probability Question 9 paper 2 .
5 unbiased coins are tossed .
Find the probability of getting 3 heads and two tails .
Here $n = 5, p = \frac{1}{2}, q = \frac{1}{2}, r = 3$ solution is $P(r) = {}^{5}_{3}C(\frac{1}{2})^{3}(\frac{1}{2})^{2} = \frac{10}{32} = \frac{5}{16}$

Or H H H T T = $\left(\frac{1}{2}\right)^{5} \times \frac{5!}{3! \times 2!}$ we find the probability of the particular case H H H T T and then multiply the result by the number of ways it can be rearranged.

If the 5 coins are tossed 8 times what is the probability of getting 3 heads and 2 tails exactly 4 times.
We know from above that when the coins are tossed once the probability of 3 heads and 2 tails is $\frac{5}{16}$ We now use binomial to find the probability $n = 8, p = \frac{5}{16}, q = \frac{11}{16}, r = 4$

$P(r) = {}^{8}_{4}C(\frac{5}{16})^{4}(\frac{11}{16})^{4} = 0.149$

2005Leaving Cert Higher maths further probability Question 9 paper 2 .
During a match john takes a number of penalty shots .The shots are independent of each other and his probability of scoring with each shot is $\frac{4}{5}$.
Find the probability that john misses his 1st 4 penalty shots.

$n = 4, p = \frac{4}{5}, q = \frac{1}{5}, r = 0$ Solution $P(r) = {}^{4}_{0}C(\frac{1}{5})^{4}(\frac{4}{5})^{0} = \frac{1}{625}$ or

(miss)(miss)(miss)(miss) = $\left(\frac{1}{5}\right)^{4} = \frac{1}{625}$

Find the probability that John Scores exactly 3 of his first 4 penalty shots
$n = 4, p = \frac{4}{5}, q = \frac{1}{5}, r = 3$ The solution is $P(r) = {}^{4}_{3}C(\frac{4}{5})^{3}(\frac{1}{5})^{2} = \frac{256}{625}$ or

Score.Score.Score.Miss= $\left(\frac{4}{5}\right)^{3} \times \left(\frac{1}{5}\right)\frac{4!}{3!} = \frac{256}{625}$

If John takes 10 penalty shots what is the probability that he scores at least 8 ?

This means the probability that he scores 8 or 9 or 10 goals . $n = 10, p = \frac{4}{5}, q = \frac{1}{5}, r = 8,9,10$

Solution $P(r)={}^{10}_{8}C(\frac{1}{5})^2(\frac{4}{5})^8 + {}^{10}_{9}C(\frac{1}{5})^1(\frac{4}{5})^9 + {}^{10}_{10}C(\frac{1}{5})^0(\frac{4}{5})^{10} = .678.$

2003Leaving Cert Higher maths further probability Question 9 paper 2 .

Whenever Anne's mobile phone rings ,the probability she answers the call is $\frac{3}{4}$.

A friend phones Anne six times .**What is the probability that she misses all the calls ?**

$n = 6, p = \frac{3}{4}, q = \frac{1}{4}, r = 0$ The solution is $P(r)={}^{6}_{0}C(\frac{1}{4})^6(\frac{3}{4})^0 = \left(\frac{1}{4}\right)^6 = \frac{1}{2096}$ or misses all the calls $\left(\frac{1}{4}\right)^6$

What is the probability that she misses the **first two** calls but answers the others.

(M = miss A = answer) ie M.M.A.A.A.A the probability of this is $\left(\frac{1}{4}\right)^2 \times \left(\frac{3}{4}\right)^4 = \frac{81}{4096}$

What is the probability that she answers exactly one call $n = 6, p = \frac{3}{4}, q = \frac{1}{4}, r = 1$

The solution $P(r)={}^{6}_{1}C(\frac{3}{4})^1(\frac{1}{4})^3 = \frac{9}{2048}$;

What is the probability that she answers at least two call?

This is the probability of 1 – [P(none)+P(one)] = $1-[{}^{6}_{1}C(\frac{3}{4})^1(\frac{1}{4})^3 + \left(\frac{1}{4}\right)^6] = \left(\frac{4077}{4096}\right)$

1996 Leaving Cert Higher maths question 9 Paper 2 :

In a game of chess against a particular opponent , the probability that Sean wins is $\frac{3}{5}$.

He plays 6 games against this opponent .What is the probability that Sean will .

(i)Lose the 2nd and 4th game and win the others?

Solution WLWLWW = $\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^2$

(ii)Win exactly 4 games $P(4)={}^{6}_{4}C(\frac{3}{5})^4(\frac{2}{5})^2 = \frac{972}{3125}$. Or WWWWLL

$\left(\frac{3}{5}\right)^4\left(\frac{2}{5}\right)^2\left(\frac{5!}{4!}\right) = \frac{972}{3125}$

(iii)Lose at least 4 games ie loses 4 or 5 or 6 games.

${}^{6}_{4}C(\frac{2}{5})^4(\frac{3}{5})^2 + {}^{6}_{5}C(\frac{2}{5})^5(\frac{3}{5})^1 + {}^{6}_{6}C(\frac{2}{5})^6(\frac{3}{5})^0 = \frac{112}{625}$

Probability Distribution table

X is the event P (x) is the probability of the event
The sum of the probabilities is always and $\sum x.(P(x)) = E(x)$ the expected value.

2004 Leaving cert Higher Maths

x	0	1	2	3
P(x)	.1	.1	.5	.3

You can be asked the mean and the standard deviation of a probability distribution table

The mean (expected value) is found using the formula $\bar{x} = \sum_{0}^{3} xP(x)$

The mean (expected value) is $(0)(.1)+(1)(.1)+(2)(.5)+(3)(.3) = 2$

The standard deviation ∂ is given by the formula $\partial^2 = \sum(x^2P(x) - (\bar{x})^2)$
(The mean of the squares – the square of the mean)
$\partial^2 = 0^2(.1)+(1)^2(1)+2^2(.5)+3^2(.3)-(2)^2 = .8 \Rightarrow \partial = \sqrt{.8} = .89$

<u>2011 Leaving cert Project maths Sample Paper Question 1 Section A</u> :

x	13	14	15	16
P(X=x)	0.383	0.575	K =(**0.038)**	0.004

(a) Complete the probability distribution.
Solution the sum of the probabilities is 1
.383 + 0.575 + k + 0.004 = 1 $\Rightarrow k = 0.038$

Calculate E(x) the expected value.

$$E(x) = \sum_{13}^{16} x.P(x) = (13)(.383) + 14(0.575) + 15(0.038) + 16(0.004) = 13.663$$

(b)If X is the age in complete years on 1st Jan 2010 of a student selected at random from all 2nd year students .explain what E(X) represents .
Solution E(X) is the **<u>mean age</u>** of a student selected at random .
10 students are selected at random find the probability that 6 were 14 years old
This is a question based on a(Bernoulli trial/ binomial distribution where).

n = 10, r = 6 , p = (0.575) , q =(0.425) (1-p) $P(6) = {}^{10}_{6}C(0.575)^6(.425)^4 = 0.248$

2009 leaving cert Higher maths further probability

x	0	1	2	3	4	5
P(x)	.01	.08	.23	.34	.26	.08

Find the mean and standard deviation

$$\bar{x} = \sum_0^5 xP(x) = (0)(.01) + (1)(.08) + (2)(.23) + (3)(.34) + (4)(.26) + (5)(.08) = 3$$

The standard deviation is given by $\partial^2 = \sum (x^2 P(x) - (\bar{x})^2)$

$$\partial^2 = 0^2(.01) + (1)^2(.08) + 2^2(.23) + 3^2(.34) + (4)^2(.26) + (5)^2(.08) - (3)^2 = 1.22 \Rightarrow \partial = \sqrt{1.22}$$

The Standard Deviation is the mean of the squares – square of the mean.

Mock Paper 2010 Trial School .

x	0	1	2
P(x)	3/28	w	5/14

The sum of the probabilities is 1 $\therefore \frac{3}{28} + w + \frac{5}{14} = 1 \Rightarrow 3 + 28w + 10 = 28 \Rightarrow w = \frac{15}{28}$

Calculate the mean use $\bar{x} = \sum_0^2 xP(x) = (0)(\frac{3}{28}) + (1)(\frac{15}{28}) + (2)(\frac{5}{14}) = \frac{43}{28}$

Sharp W531 Calculator and Statistics

The sharp w531 calculator can be used to find the **mean and standard deviation** as follows.

X	1	2	3	4
f	5	7	6	2

Method

(i) Press mode then pres 0 the screen reads STAT 0 (SD).

(ii) Input data as follows 1 (x,y) 5 Data (change) the screen reads DATA SET = 1 .

(iii)Then 2(x,y) 7 Data etc the screen will read DATA SET =2

Continue to input data when all the data has been entered the screen will show DATA SET = 4.

To find the **mean** press RCL 4 ($\bar{x}$) = 2.25

To find the standard deviation press RCL 6 (δx) = 0.94207

To Find the **Correlation coefficient** from a scatter plot usinf Sharp w 531 calculator.

X Score (independent variable)	Y score (Dependent Variable)
5	20
8	18
6	22
7	28
10	27

Method :

(i)Press mode then press 1 SCREEN READS STAT 1 (LINE).

Input data as follows 5(x,y) 20 data screen reads data set = 1 .

Repeat for the rest of the data .After the last imput screen should read DATA SET = 5.

To find the correlation coefficient press RCL ÷ (DIVISION SYMBOL) =0.417

To find the equation of the regression line (line of best fit)

To find the slope of the regression line pres RCL) (RIGHT BRACKET) = .946
To find where the regression line cuts the y axis pres RCL ((LEFT BRACKET) = 16.189
Therefore the equation of the line of best fit is $y-16.189=.946x$ (use $y-y_1=m(x-x_1)$

Statistics Main points

Statistics is the science of collecting and analysing Data .
Data means numerical facts or non –numerical facts or information .
The syllabus states we must be able to deal with many types of Data (see notes on types of Data)
Catagorical (male ,female) :**Discrete** Number of siblings: **Continuous**: Distance from School
Ranked variable is a categorical variable for which the categories imply some order or relative position.
Population : A population is a set of measurements or counts about which we want to draw a conclusion . A population is similiar to a Universal set .
Sample : A sample is a subset of a population or a set of some of the measurements or counts from a population .

Mean : The is the average, the mean of a set of Data is given by the formula $\dfrac{\sum x}{\sum n}$

The Mean of the set 1,2,3,4,4,10,11 is $\dfrac{1+2+4+4+5+10+11}{7}=5$
The Median :This is the middle number when the numbers are arranged in order of size .In the case above there are 7 numbers the 4th number is the median the median is 4.
The Mode is the number that occurs most often in this case the mode is 4.

The mean of a frequency table is $\dfrac{\sum fx}{\sum f}$

The mean of the frequency table

x	1	2	3	4
f	3	4	2	1

$=\dfrac{(1)(3)+(2)(4)+(3)(2)+(4)1}{3+4+2+1}=\dfrac{21}{10}=2.1$

The Median is the sum of the two middle numbers divided by 2 $=\dfrac{2+2}{2}=2$ **The Mode is 2 .**

The Standard Deviation; This is a measure of dispersion(spread) around the mean
The easiest definition of the standard deviation is
The standard deviation is the square root of the mean of the squares of the data minus the mean squared
The standard deviation of set 1,2,3,4,4,10,11 is

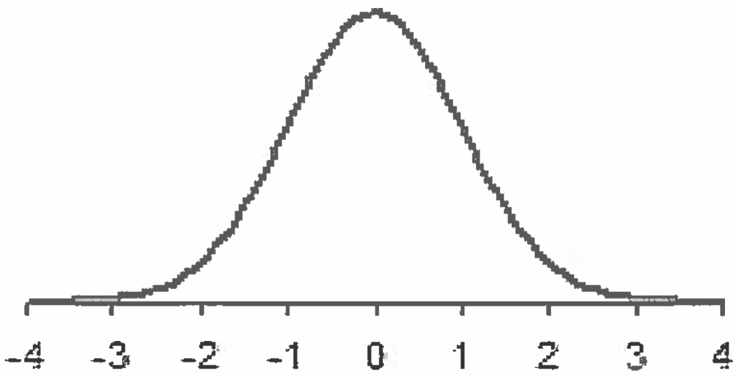
$$\delta = \sqrt{\frac{1^2 + 2^2 + 3^2 + 4^2 + 4^2 + 10^2 + 11^2}{7} - (5)^2} = 3.625$$

x	1	2	3	4
f	3	4	2	1

The standard deviation from the frequency table above is

$$\delta = \sqrt{\frac{(1)^2(3) + (2)^2(4) + (3)^2(2) + (4)^2(1)}{10} - (2.1)^2} = .943$$

Normal Distribution (Bell shaped curve)



Properties of this Bell shaped curve
The mean is 0 and the z axis is in units of the standard deviation ie 1 = 1 standard deviation from the mean .
Approximately 68% of the area under any normal distribution curve lies within one standard deviation of the mean .
Approximately 95% of the area under any normal distribution curve lies within two standard deviations of the mean
Approximately 99.7% of the area under any normal distribution curve lies within three standard deviations of the mean.

Example1 we are told that the heights of male students in University in Ireland is normally distributed with a mean $\mu = 168$ cm and a standard deviation $\delta = 6cm$ we can state that of males , 95% have heights between $168 \pm 2(6) = 156 - 180$ This is equivalent to saying that the probability that randomly selected male student will have a height between 156cm and 180cm cm is 0.95 .

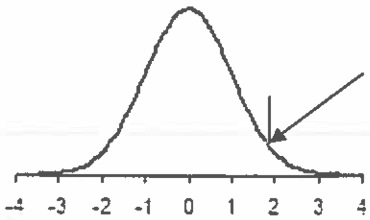
When given information regarding normally distributed data in order to use the Normal; distribution tables supplied in the exam .We must transform our normal distribution into one where the mean $\mu = 0$ and the standard deviation $\delta = 1$.

We do this by calculating "z" values where z is given by the formula $z = \frac{(x-\mu)}{\delta}$

Example 2: What is the probability that a male student has a height less than 176cm .This is $P(x) < 176cm \rightarrow P(z) < \frac{(176-168)}{6} < 1.33$ **now look up 1.33 in the tables to get 0.9082**

Example 3: What is the probability that a student has a height greater that 180cm?

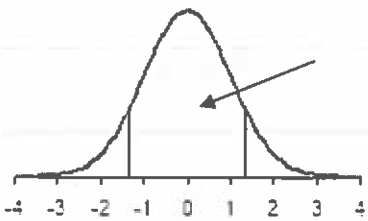
We are looking for $P(x) > 180cm \rightarrow$
$P(z) > \frac{(180-168)}{6} > 2$



We are looking at the area to the right of 2 .We can say

$x + y = 1 \Rightarrow x = 1 - y = 1 - P(2) = 1 - .9772 = 0.0228$

What is the probability that the heights lie between 162cm and 174cm? We are looking for
$163 \le P(x) \le 174 \rightarrow P(\frac{162-168}{6} \le z \le \frac{174-168}{6}) \Rightarrow -1 \le p(z) \le 1$



From the diagram we can see that
$x + 2y = 1 \Rightarrow x = 1 - 2y \Rightarrow x = 1 - 2(1-y)$
$x = 2y - 1 \Rightarrow x = 2P(1) - 1 = 2(0.8413) - 1 =).6826$

<u>Leaving Cert Higher maths question 9 (further probability)</u>

The 95% confidence interval for a sample mean $\left(\bar{x}\right)$ is $\left(\bar{x}\right) \pm 1.96\left(\frac{\delta}{\sqrt{n}}\right)$ this is the method was used pre 2010-2014 project maths during the transition period and will be examined from 2015 onwards. The formula for the margin of error of a sample proportion is $\frac{1}{\sqrt{n}}$ .this was used on the LCH from 2010-2014 and is now on the ordinary level.

Question 1 LC2010: A bakery produces muffins .A random sample of 50 muffins is selected and weighed. The mean of the sample is 80grams and the standard deviation is 6 grams .

Note the mean of a population and the mean of the sample are the same but the standard deviation of a sample is $\frac{\delta}{\sqrt{n}}$.

Form a 95% confidence interval for the mean weight of a muffin produced by the bakery.
This means that 95% of the sample means lie between

$\bar{x}+1.96\frac{\delta}{\sqrt{n}}$ and $\bar{x}+1.96\frac{\delta}{\sqrt{n}}$ (between 2 standard deviations of the mean)

$80-1.96\frac{6}{\sqrt{50}}=78.3$ and $80+1.96\frac{6}{\sqrt{50}}=81.66$ The 95% confidence interval is {78.3,81.66}

<u>Question 2</u>: LC 2009.A coin is slightly bent and is thought to favour heads .The coin is tossed 100 times to test the null hypothesis that it is favours heads .In the experiment we get 55 heads . Show that this result is not significant at the 5% level .
If the probability of getting 55 heads out of 100 tosses lies between ± 1.96 (2 standard deviations of the mean) the result is not significant.

We know $P(z)=\frac{x-\bar{x}}{\delta}$ we need to find $\bar{x}$ and δ for a binomial distribution (the result of carrying out a series of n Bernoulli trials) The mean $\bar{x}=np, \delta=\sqrt{npq}$

N= 100, $p=\frac{1}{2}, q=\frac{1}{2}$ $\bar{x}=100(\frac{1}{2},)=50, \delta=\sqrt{100(\frac{1}{2})(\frac{1}{2})}=5$

$\therefore P(z)=\frac{55-50}{5}=1$ look up 1 in the tables =0.8413 < 0.95.

Therefore the result is not significant at 5% level
How many times should the coin be tossed in order that an observation of 95% heads would be regarded as significant at a 5% level?
This time we are looking for n.

We know $P(z)=\frac{x-\bar{x}}{\delta}$, $\bar{x}=np=n\frac{1}{2}, \delta=\sqrt{n(\frac{1}{2}\frac{1}{2})}=\frac{\sqrt{n}}{4}$.

$\therefore P(z)=\frac{.55n-\frac{n}{2}}{\frac{\sqrt{n}}{2}}=\frac{(1.1)n-n}{\sqrt{n}}=0.1\sqrt{n}$ this must have a probability greater than 0.95

(P(1.65) = 0.95 Therefore $0.1\sqrt{n}>1.65 \Rightarrow \sqrt{n}>16.45 \Rightarrow n>(16.5)^2=272.25$,n = 272 times.

<u>Question 3 LC2008</u>: In order to test if a coin is not biased it is tossed 400 times .The number of heads is X .Between what limits should X lie in order that the hypothesis not be rejected at 5% significant level.

The question being asked here is what is the range of values of X that give a result between ± 1.96. Like the last question the mean $\bar{x} = np = 400\frac{1}{2} = 200, \delta = \sqrt{400(\frac{1}{2}\frac{1}{2})} = 10$

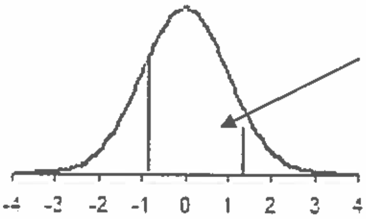
$$-1.96 \le \frac{x - \bar{200}}{10} \le 1.96 \Rightarrow -19.6 \le x - 200 \le 19.6 \Rightarrow 180.4 \le x \le 219.6$$

$$\Rightarrow 180 \le x \le 219$$

<u>Question 4</u> LC2007: The amounts due on monthly phone bill in Ireland are normally distributed with a mean of €53 and a standard deviation of €15.
If a bill is chosen at rand what is the probability that the amount due is between €47 and €74

We find $P(47 \le x \le 74)$ now change to standard units using $P(z) = \frac{x - \bar{x}}{\delta}$ this gives

$\frac{47 - \bar{53}}{15} \le z \le \frac{74 - 53}{15} \Rightarrow -0.4 \le z \le 1.4$ Now use the Bell shaped curve to find the region



From the diagram we can see

$x + y + z = 1 \Rightarrow x = 1 - (y + z)$

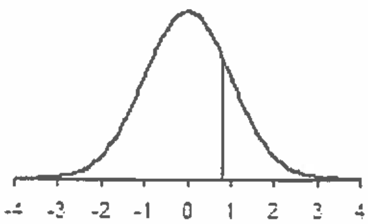
$y = 1 - P(y), z = 1 - P(z) \Rightarrow$

$x = P(y) + p(z) - 1 = P(0.4) + p(1.4) - 1 = 0.5746$

A random sample of 900 phones is taken find the probability that the mean of the random sample is greater than 53..3.
We are looking for the probability

$P(x > 53.3) \Rightarrow P(z > \frac{53.3 - \bar{53}}{\frac{15}{\sqrt{900}}}{**} \Longrightarrow P(z > 0.6)$ ** must us the standard deviation of the sample

Again we can use our bell shaped curve to find P(>0.6)



$x + y = 1 \Rightarrow x = 1 - (y)$

$y = 1 - P(y) = 1 - P(0.6) = 1 - 0.7257$

$x = 0.2743$

Question 5:2006 LCH 2006; The marks awarded in an exam are normally distributed with a mean mark of 60 and a standard deviation of 10.

A sample of 50 students has a mean mark of 63.
Test at the 5% level of significance the hypothesis that this is a random sample from the population..

Given $\bar{x} = 60, \delta = 10$ therefore the standard deviation of the sample is $\frac{\delta}{\sqrt{n}} = \frac{10}{\sqrt{50}} = \delta_1$

$$\frac{x-\bar{x}}{\delta_1} = \frac{63-60}{\frac{10}{\sqrt{50}}} = \frac{3}{10}\sqrt{50} = 2.123 > 1.96$$ this is not a random sample.

Question 6 LCH2011:
The mean percentage mark for candidates in 2010 LC Higher maths was 67% and a standard deviation of 10.4%.The suggestion that candidates who appealed their results have on average similiar results to all other candidates. A random sample of candidates who appealed was taken .The mean mark of the sample was 69.3%.Show that if the sample size was 25,then the result is not significant at the %5 level .

Given $n = 25, \bar{x} = 67, \delta = 10.4$ therefore the standard deviation of the sample is

$$\frac{\delta}{\sqrt{n}} = \frac{10.4}{\sqrt{25}} = 2.08 = \delta_1 \quad \frac{x-\bar{x}}{\delta_1} = \frac{69.5-67}{2.08} = \frac{2.5}{2.08} = 1.105 < 1.96$$ result is not significant.

Show if the sample size is 100 that the result is significant at the 5% level

$$\frac{x-\bar{x}}{\delta_1} = \frac{69.3-67}{\frac{10.4}{\sqrt{100}}} = \frac{2.5}{10.4}(10) = \frac{2.3}{1.04} = 2.211 > 1.96$$ The result is significant

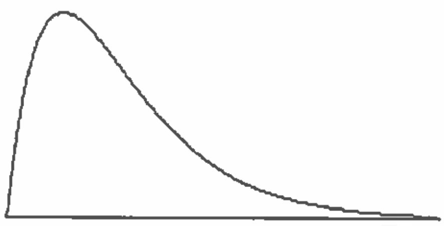
What is the smallest sample size for which this result could be regarded as significant at a 5% level.

$\mu = 67, \delta = 10.4, 69.3, n = n$ we are being asked to find n for which the

$$\frac{x-\bar{x}}{\delta_1} \geq 1.96 \Rightarrow \frac{69.3-67}{\frac{10.4}{\sqrt{n}}} \geq 1.96 \Rightarrow \frac{(2.3)\sqrt{n}}{10.4} \geq 1.96 \Rightarrow \sqrt{n} \geq 8.862 \Rightarrow n > 79.55 \Rightarrow n = 79$$

Comment this type of question is very similiar to the project maths probability/statistics.

2012 Sample Paper Question 7b



The distribution of the hourly earnings of all employees in Ireland in October 2009 is shown in the diagram. It can be seen that the distribution is positively skewed.
The mean is €22.05.
The median is €17.82.
The standard deviation is €10.64.
The lower quartile is €12.80.
The upper quartile is €26.05

(i)If six employees are selected at random from this population what is the probability that exactly four of them had hourly earnings of more than €12.80.

(i)The lower quartile is €12.80 that means that 25% of the employees earn €12.80 or less .This means that the probability of earning €12.80 or less is 0.25 .The probability of earning more than €12.80 is 0.75.

We use ${}^{n}_{r}C(p)^{n-r}(q)^{r}\, n = 6, r = 4, p = 0.75, q = 0.25 \Rightarrow {}^{6}_{4}C(.75)^{4}(.25)^{2} = 0.2966$

In a computer simulation, random samples of size 200 are repeatedly selected from this population and the mean of each sample is recorded. A thousand such sample means are recorded.

(ii) Describe the expected distribution of these sample means. Your description should refer to the shape of the distribution and to its mean and standard deviation.

(ii)The expected distribution of these sample means will be approximately normal (because of the large sample size) even thought the parent distribution was not normal. The mean of the sample means will be €22.05. the standard deviation will be $(€10.64) \div \sqrt{200}$.

(iii) How many of the sample means would you expect to be greater than €23?
Because the distribution of the sample means is normal we can use

$$P(z) = \frac{x - \bar{x}}{\delta_1}, \bar{x} = €22.05, x = €23, \delta_1 = \frac{€10.64}{\sqrt{200}} = 0.7524 \Rightarrow p(z) > €23 =$$

$$\frac{23 - 22.05}{0.7524} = \frac{0.95}{0.7524} = 1.26 \Rightarrow P(z) > 1.26 \Rightarrow 1 - P(1.26) = 1 - .8962 = 0.1038$$

Therefore the number of sample means greater than €23 is $1000(.1038) = 103$

This question was added to the 2010 sample to account for the increase in marks from 50-75.

The Central Limit Theorem

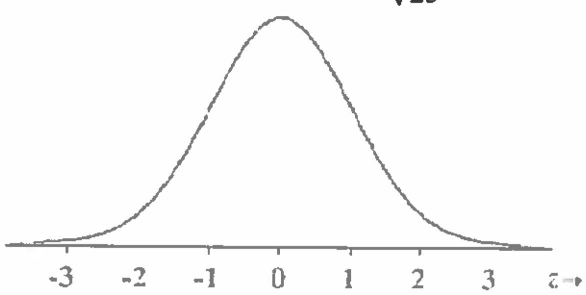
When we sample a population the numerical value of the mean of the sample is a random variable $\bar{X}$ that estimates the population mean .The value of the sample $\bar{X}$ will vary because the sample data varies from sample to sample the mean of the samples $\bar{X}$ is itself a random variable and has a probability distribution. This distribution is called the sampling distribution of $\bar{X}$ .It is the distribution of all the means of all the possible samples of a given size that can be drawn from the population.
The central limit theorem states .When sampling repeatedly from a population which has a mean μ and a standard deviation δ using a fixed sample size n.

The sampling distribution of $\bar{X}$ (the means) will approach a normal distribution with mean μ and a standard deviation $\frac{\delta}{\sqrt{n}}$ (this **is called the standard error of** $\bar{X}$).
The central limit states that if n is large enough ($n \geq 30$) the distribution of the sample means is approximately normal regardless of the distribution of the population.
The mean of the sample means is μ (same as the population) and the standard deviation of the sample means is $\frac{\delta}{\sqrt{n}}$.

Example 1: A random sample of size n = 49 is taken from a population with $\mu = 75$ and a standard deviation of $\delta = 14$ .Find the probability that the mean of the sample is greater than 78.93

$$P(\bar{X}) > 78.93 \rightarrow P(Z) > \frac{\bar{x}-\mu}{\frac{\delta}{\sqrt{n}}} = \frac{78.93-75}{\frac{14}{\sqrt{49}}} = \frac{3.93}{2} = 1.96 \rightarrow P(Z) > 1.96 = 1-0.9750 = 0.025$$

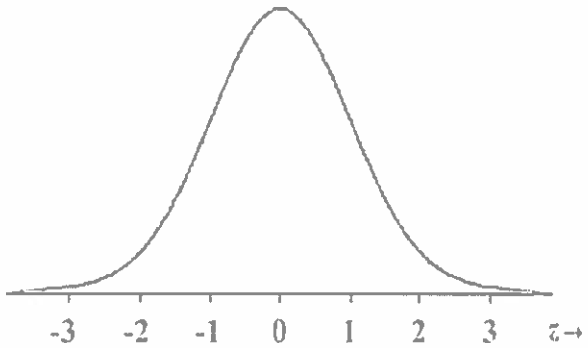
Example 2: IQ scores of adults are normally distributed
A sample of 25 adults is taken from a population of mean $\mu = 100$ and a standard deviation $\delta = 15$ .find the probability that the mean IQ of the sample is less than or equal to 94. $P(\bar{X}) \leq 94 \rightarrow P(Z) \leq \frac{94-100}{\frac{15}{\sqrt{25}}} \rightarrow P(Z) \leq \frac{-6}{3} \rightarrow P(Z) \leq -2$



$P(Z) \leq -2 = 1 - P(2) = 1 - 0.9772$
=).0228

Example 3: The lifetime of a certain type of bulb is normally distributed with a mean of 480 hours and a standard deviation of 40 hours .A sample of 10 bulbs is taken. What is the probability that the mean lifetime of the sample of 10 bulbs is at least 500 hours?

We want to find $P(\bar{X}) \geq 500 \rightarrow P(Z) \geq \dfrac{500-480}{\dfrac{40}{\sqrt{10}}} = P(Z) \geq 1.58$



$P(Z) \geq 1.58 \Rightarrow P(Z) \geq 1 - P(1.58) =$
$1 - 0.9429 = 0.0571$

Confidence Interval

Confidence Interval is a range of values believed to contain a known population parameter. For leaving cert we deal with a 95% level of confidence that the interval does indeed contain the parameter of interest.

For a 95% confidence interval the critical value is 1.96

The 95% confidence interval is usually written as $\bar{x} \pm 1.96\dfrac{\delta}{\sqrt{n}}$.

Example: A bank wants to estimate the amount of money people have in their accounts a random sample of 100 accounts is taken the mean $\bar{x} = €584$ the standard deviation is known to be €120 .Construct a 95% confidence interval for the mean amount of money people have in their accounts in the bank.

$\bar{x} \pm 1.96\dfrac{\delta}{\sqrt{n}} = 584 \pm 1.96\dfrac{120}{\sqrt{100}} = 584 \pm 1.96(12) = €560.48,.€607.52$ .The bank can be 95% confident that the amount of money in the accounts in the bank lies between €560.48 and €607.52

5% Test of significance $z \geq 1.96$ if z has a normal distribution.

Example

A confectionary company claims that the minimum weight of each box of chocolates is 350g.The weights of the chocolates in each box are normally distributed with a mean equal to a specific weight and a standard deviation of 8g.

A sample of 50 boxes are selected at random and the mean weight of the sample boxes is 348g .Use a hypothesis at the 5% significant level to determine whether there is sufficient evidence to support the company's claim .state the null hypothesis Ho (status quo)

Ho = 350 .Now change the new mean into a z score $\dfrac{348-350}{\dfrac{8}{\sqrt{50}}} = -1.7677$

$-1.96 \leq -1.7677 \leq 1.96$ Therefore the result is not significant

Hypothesis testing The null Hypothesis Ho ;

If we want to decide if a coin is loaded we formulate the hypothesis that the coin is fair (ie P = 0.5 where p is the probability of heads) or if we want to find out if one procedure is better than another we can decide there is no difference between the procedures (any differences may be caused by fluctuations in the sampling. Such hypotheses are called Ho the null hypothesis .In general for leaving cert the Ho is usually the status quo.

Note H_0 =(is always equals)

The Alternative Hypothesis from above if the null hypothesis is p = 0.5 the alternative hypothesis might be p > 0.5, p < 0.5 $p \neq 0.5$.The alternative hypothesis is denoted by Hi.

Note H_1, is never equals, but is always one of the following $<, \neq, >$.

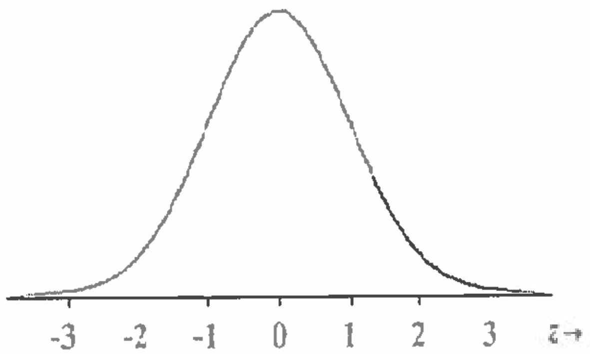
When testing a given Hypothesis ; If we suppose a particular Hypothesis is true but find that the results from a random sample differ greatly from the results expected under the hypothesis (expected on the basis of pure random chance based sampling theory) .If we got 15 heads when tossing a coin 20 times, we would be inclined to reject the coin to be a fair coin Hypothesis tests are procedures that help us determine whether the results from samples differ significantly from the results expected and as a result help us decide whether to accept or reject the hypothesis.

Type 1 Error; this is where we reject a hypothesis when it should be accepted.

Level of Significance; in testing a particular hypothesis the maximum probability we would be willing to risk a Type 1 error is called the level of significance. For the leaving cert we deal with a **5% level of significance**. When using the 5% level of significance then there are 5 chances in 100 that we would reject the hypothesis when we should have accepted it .Or we are **95% confident** that we have made the correct decision.

Hypothesis testing and normal distributions;

If the z score of the actual sample statistic lies between -1.96 and $+1.96$ we can be 95% confident that the hypothesis is true .If the z score of the actual sample does not lie between -1.96 and $+1.96$ we conclude that the event can only happen with a probability of 0.05 if the given hypothesis is true. We would say that the Z score differs significantly from what would be expected under the hypothesis and we would be inclined to reject the hypothesis.



The set of z scores outside the range -1.96 and $+1.96$ is called the critical region of the hypothesis.

We reject the hypothesis at the 5% level of significance if the score of the statistic is outside the range -1.96 and $+1.96$ ..

We accept that the hypothesis is correct if the z score of the statistic lies in the range -1.96 and $+1.96$

2001 Leaving H Question 9 paper 2 . A particular drug gives relief from pain. The period of pain relief reported by people who are treated with the drug is normally distributed with mean 50 hours and standard deviation 16 hours.

In a random sample of 64 people who have been treated with the drug, what is the probability that the mean period of pain relief reported is between 48 hours and 53 hours?

Solution

We are given that the mean of the population is 50 hours and that the standard deviation of the population is 16 hours .We are being asked to find the probability that the mean of a sample of 64 people will lie between 48 and 53 hours . We are being asked for

$$P(48 \le x \le 53) \to P(\frac{48-50}{\frac{16}{\sqrt{64}}} \le z \le \frac{53-50}{\frac{16}{\sqrt{64}}}) \to P(-1 \le z \le 1.5)$$

$$= P(1.5) - P(-1) = P(1.5) - (1 - P(1)) = P(1.5) + P(1) - 1 = 0.9332 + 0.8413 - 1 = 0.7745$$

Confidence Interval for a sample

1996 Question 9c

A company installs a new machine in a factory. The company claims that the machine will fill bags with sugar having a mean mass of 500g and a standard deviation of 13g . 36 bags are checked in a random sample. Their mean mass is 505g.At the 5% level of signifance is the result consistent with the company's claim

1996 LC H Question 9 paper 2

We are given the mean of a population as 500g ,the standard deviation of the population is 18g .We are asked if the mean of 505g found from a sample of a sample of 36 bags is this result consistent at the 5% level of confidence with the company's claim ?

Solution get the z score of the sample and if it is less than 1.96 the result is not significant.

$$z = \frac{x - \bar{x}}{\frac{\delta}{\sqrt{n}}} = \frac{505-500}{\frac{18}{\sqrt{36}}} = 1.67 < 1.96$$ The result is not significant at the 5% level and is consistent with the company's claim.

1998LCHQ9c

A car manufacturing company tested a sample of 150 cars of the same model to estimate the mean number of kilometres travelled per litre of petrol for all cars of that model. The sample mean of kilometres travelled per litre of petrol consumed was 13.52 and the standard deviation was 2.23.

Form a 95% confidence interval for the mean numbers of kilometres travelled per litre of petrol consumed for all cars of that model Give calculations correct to two places of decimals.

1998 LCH Q9(b) paper 2,Solution

We are given the sample mean as 13.52 m (same as mean of the population).The standard deviation of the population was 2.23 .The size of the sample is 150 .

To calculate the confidence interval we find the interval

$$\bar{x} \pm 1.96\left(\frac{\delta}{\sqrt{n}}\right) = 13.52 \pm (1.96)\frac{2.23}{\sqrt{150}} = 13.16, 13.88$$

1999Q9c.

The lifetime of a particular bulb is normally distributed with a mean of 1500 hours and a standard deviation of 120 hours.If 140 bulbs are purchased, how many can be expected to have a lifetime between 1400 and 1730 hours inclusive . Give calculations correct to two places of decimals.

1999 LCH Q9(c) paper 2. Solution

Given the mean of the population is 1500 hours and the standard deviation of 120 hours. We are looking for the probability

$$P(1400 \le x \le 1730) \rightarrow P\left(\frac{1400-1500}{120} \le z \le \frac{1730-1500}{120}\right) \rightarrow P(-0.83 \le z \le 1.92)$$

$$= P(1.92) - P(-0.83) = P(1.92) - (1 - P(0.83)) = 0.9726 + 0.7967 - 1 = 0.7693$$

The number out of 140 that will be accepted $= 140x0.7693 = 107.7 = 108$

P values to be examined from 2015 on .

P values were added to the syllabus for 2015 .Section 1.7 of the syllabus states Perform univariate large sample tests of the population mean (two-tailed z-test only) use and interpret p-values. What exactly is required by the SEC was clarified by the NCCA as follows"*Use and interpret are the extent of the requirement in relation to p-values. From Section 1.3 (LCHL) students are already expected to be able to read probabilities from the standard normal distribution table*".

So what are p values .We need to review the following terms.

Null Hypothesis H_0(status quo) this assumes no difference .

Alternative Hypothesis H_1 this assumes a difference.

P Value : A decision between the two hypothesis is made by viewing the 'p Value' which is the probability of the collected value or a more extreme event happening under the assumption of the null hypothesis.

If this probability is **small** H_0 is **rejected** in favour of H_1 "If the p is low null has to go"!

If this probability is large H_0 is **accepted** in favour of H_1 "if the p is high the null will fly".

What is small? If $p \leq 0.05$ H_0 is **rejected** in favour of H_1

If $p > 0.05$ H_0 is not **rejected.**

The p-value can also be interpreted as follows

P-Value of Test statistic	Statistically Significant	Formal Action	In English
Greater than 0.1	No	Do not reject H_0	No evidence to reject H_0
Between 0.1 and 0.05	No	Do not reject H_0	Weak evidence to reject H_0
Between 0.05 and 0.01	Yes at 95% level	Reject H_0 at 95% confidence interval	Evidence to reject H_0

How do we calculate P values ?

The Questions on the Leaving Cert H are confined to univariate large sample tests of the population mean (two-tailed z-test only) at the 5% significant level .

The method is as follows

(i)Decide H_0= k,(ii)Decide $H_1 \neq$ k.(iii)Decide significance level (5% = 0.05).

(iv)Find the test statistic using $\frac{x-\bar{x}}{\frac{\delta}{\sqrt{n}}}$ =z , (v)look up P(z) in the tables (vi)Find $1-(P(z))$ and multiply by 2.

This is the p value make your decision based on the table above.

Example 1:
A machine is designed to produce rods with a mean of 2cm and a standard deviation of 0.02cm. The lengths of the rods are normally distributed.
The machine is moved to a new factory .In order to check whether the mean length has been affected by the move. A sample of 10 rods are measured , the standard deviation is assumed to be unchanged .If the lengths of the 10 rods are given below test at the 5% significance level whether the setting has altered .
2.04,1.97,1.99,2.03,2.04,2.10,2.01,1.98,1.97,2.02 .
The new mean is 2.015.

The H_0 (null hypothesis) is mean is 2cm .

The H_1 (alternative hypothesis) mean $H_1 \neq$ 2cm

This is a two tailed test we are looking for $P(x) \geq 2.015$ or $P(x) \leq -2.015$

We find the test statistic $P(z) \geq \dfrac{2.015-2}{\dfrac{0.02}{\sqrt{10}}} = 2.372 \rightarrow P(z) \geq 2.378$

Now look up 2.378 in the standard normal tables =0.9913 this is the probability that $P(z) < 2.378 \rightarrow P(z) \geq 2.378 = 1-0.9913 =).0087$ this is the area to the right of 2.372 the total area which applies is 2 x (0.0087) =0.0174 (this is the P-value) .
The reason we multiply by 2 is because $P(x) \geq 2.015$ or $P(x) \leq 2.015$.
Since the p value is 1.74% which is less than 5% or $0.017 < 0.05$.
From the table above
We reject the null hypothesis at 95% confidence level.

Example 2: Metal struts have a specified mean of 2.855m The lengths have a normal distribution with standard deviation of 0.0352m.A sample of 15 struts is measured and has a mean of 2.841m .A test is to be carried out at the 5% significance level to decide whether the batch is from the specified population .
State the null hypothesis and the alternative hypothesis and calculate the P-value .
State the conclusion of the test.

The H_0 =2.855.

The $H_1 \neq$ 2.855cm

We are looking for the both $-2.841 \geq P(x) \cup P(x) \geq 2.841$ now change to Z.

$P(z) \leq \dfrac{2.2841-2.855}{\dfrac{0.0352}{\sqrt{15}}} \rightarrow P(z) \leq -1.54$ look up 1.54 in the standard normal tables =0.9382

$P(z) \leq -1.54$ this is the area to the left of -1.54 (which is the same as the area to the right of 1.54) = $1-0.9382 = 0.0618$ we need the sum of the areas to the left of -1.54 and the right of 1.54 = $2 \times (0.0618) = 0.123$ this is the p-value . $0.123 > .05$ from the table above we do not reject the null hypothesis at the 5% level of significance.

Another way to do the following question using p- values .

Example;3
A confectionary company claims that the minimum weight of each box of chocolates is 350g.The weights of the chocolates in each box are normally distributed with a mean equal to a specific weight and a standard deviation of 8g.
A sample of 50 boxes are selected at random and the mean weight of the sample boxes is 348g .Use a hypothesis at the 5% significant level to determine whether there is sufficient evidence to support the company's claim .state the null hypothesis Ho (status quo)
Ho = 350 .Now change the new mean into a z score $\frac{348-350}{\frac{8}{\sqrt{50}}}=-1.7677$

$-1.96 \leq -1.7677 \leq 1.96$ Therefore the result is not significant

P- Value method!

Take H_0=350 and H_1≠ 350

We find the z score $\frac{348-350}{\frac{8}{\sqrt{50}}}=-1.7677$ We need to find the probability that

$-1.7677 > z \cup z > 1.7667$ (We will use z = 1.77))

From the tables we get $P(z) < 1.77 = 0.9616$.
Now find $P(z) \geq 1.77 = 1-0.9616 = 0.0384$ multiply by 2 to get 0.0768.
$0.0768 > 0.05$ from the table above we have no reason to reject H_0.
The result is not significant at the 5% level.

Example :4 The alkalinity of soil is measured by its pH value .It has been found from many previous measurements that the mean pH from a particular area is 8.42 and a standard deviation of 0.74.After a very hot summer a sample of 36 measurements of the pH were taken it was found that the sample mean is 8.63, the standard deviation is unchanged .
Calculate the p value of the test of the hypothesis that the mean pH is greater than 8.42.
$H_0 = 8.42$, $H_1 > 8.42$

Find the z score we need $P(z) > \frac{8.63-8.42}{\frac{0.74}{\sqrt{36}}} > 1.703$ we need to find the probability that

$z > 1.703$ Look up 1.703 in the tables to get $P(z) \leq 1.703 = 0.9554$.
$\rightarrow P(z) > 1.703 = 1-0.9554 = 0.0446 < 0.05$ we therefore reject H_0 and conclude that the pH is greater than .This is a one tailed test (we expect only two tailed tests to be examined in the leaving cert!)

Questions base on proportion (no standard Deviation given)

Example

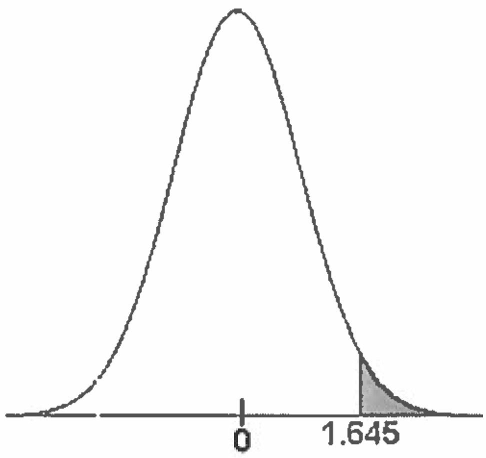
1500 randomly selected pine trees were tested for traces of the Bark Beetle infestation. It was found that 153 of the trees showed such traces. Test the hypothesis that more than 10% of the Tahoe trees have been infested. (Use a 5% level of significance)

Solution

The hypothesis is

H_0: $p = .1$

H_1: $p > .1$



We have that

$$\hat{p} = \frac{153}{1500} = .102$$

Next we compute the z-score

$$z = \frac{0.102-0.1}{\sqrt{0.1(1-0.1)/1500}} = 0.26$$

Since we are using a 95% level of significance with a one tailed test, we have $z_c = 1.645$. The rejection region is shown in the picture. We see that 0.26 does not lie in the rejection region, hence we fail to reject the null hypothesis. We say that there is insufficient evidence to make a conclusion about the percentage of infested pines being greater than 10%.

Exercises

A. If 40% of the nation is registered republican. Does the Tahoe environment reflect the national proportion? Test the hypothesis that Tahoe residents differ from the rest of the nation in their affiliation, if of 200 locals surveyed, 75 are registered republican.

B. If 10% of California residents are vegetarians, test the hypothesis that people who gamble are less likely to be vegetarians. If the 120 people polled, 10 claimed to be a vegetarian.

Example 1:About 10% of the human population is left-handed. Suppose a researcher at Trinity College speculates that students in the College of Arts and Architecture are more likely to be left-handed that people in the general population. A random sample of 100 students in the College of Arts and Architecture is obtained such that 18 were found to be left-handed.
Research Question: *Are artists more likely to be left-handed than people in the general population?*

Example 2 :
In an election a campaign manager claims that the percentage who prefer his candidate is 60% . A polling company conducts a random survey of 500 registered voters to test if the managers claim is correct, he finds that 58% of the sample will vote for the managers candidate .Perform a test at the 5% level to see if the managers claim is correct.

Example 3 : In a survey of 1000 students 220 said they were smokers .Before this survey the health service claimed the national percentage is 19.8%.At the 5% level does the data in the survey support the health service claim?

Example 4: In a poll of 1020 people 70% of those polled said they visited Temple Bar . Calculate the 95% confidence interval to estimate the population proportion who had visited Temple Bar .

Question 5:2006 LCH 2006; The marks awarded in an exam are normally distributed with a mean mark of 60 and a standard deviation of 10.
A sample of 50 students has a mean mark of 63.
Test at the 5% level of significance the hypothesis that this is a random sample from the population..

P Values

Example 1:
A machine is designed to produce rods with a mean of 2cm and a standard deviation of 0.02cm.
The lengths of the rods are normally distributed.
The machine is moved to a new factory .In order to check whether the mean length has been affected by the move. A sample of 10 rods are measured , the standard deviation is assumed to be unchanged .If the lengths of the 10 rods are given below test at the 5% significance level whether the setting has altered .2.04,1.97,1.99,2.03,2.04,2.10,2.01,1.98,1.97,2.02 .
Calculate the P value

__Example 2:__ Metal struts have a specified mean of 2.855m The lengths have a normal distribution with standard deviation of 0.0352m.A sample of 15 struts is measured and has a mean of 2.841m .A test is to be carried out at the 5% significance level to decide whether the batch is from the specified population .
State the null hypothesis and the alternative hypothesis and calculate the P-value .
State the conclusion of the test.
Calculate the P value

Financial Maths
Essentials

The Compound Interest formula $A = P(1+\frac{R}{100})^n$ where P is the principle, R is the percentage rate ,n is the time (number of years or number of months)
gives the result of investing €P for n years at R% .A = Principle + Interest

Example 1: Find what €5000 will amount to in 8 years at 6%.

Answer $A = P(1+\frac{R}{100})^n$ P = €5000, R = 6%, n = 8 .

A = $€5000(1+\frac{6}{100})^8$ = €7969.24

The Interest earned is €7969.24-€5000 = €2969.24.

Example 2: If €4000 becomes €4434.8715 in 3 years find the % rate

$A = P(1+\frac{R}{100})^n$ A = €4434.8715, P = €40000

$€4000(1+\frac{R}{100})^3 = €4434.8715 \Rightarrow (1+\frac{R}{100})^3 = \frac{4434.8715}{4000} = 1.108717875$

$\Rightarrow 1+\frac{R}{100} = \sqrt[3]{1.108717873} = 1.035 = R = 3.5\%$

Example 3 **Leaving Cert ord Level 2011 question 2 paper 1.**

Question 2 **(25 marks)**

(a) A certain deposit account will earn 3% interest in the first year and 6% interest in the second year. The interest is added to the account at the end of each year. If a person invests €20 000 in this account, how much will they have in the account at the end of the two years?

(b) Show that, to the nearest euro, the same amount of interest is earned by investing the money for two years in an account that pays compound interest at 4·49% (AER).

(a) Use the compound interest formula $A = P(1+\frac{R}{100})^n$

Year 1:

P = €20,000 , R= 3% , n = 1 : $A = 20000(1+\frac{3}{100})^1$ = €20600

Year 2;

P = €20600,R = 6%, n = 1 $A = 20600(1+\frac{6}{100})^1$ = €21836

Interest earned €21836-€20,000 = €1836

(b)If €20000 is invested for 2 years @ 4.49% it amounts to

$A = 20000(1+\frac{4.49}{100})^2$ = €21836.32. Therefore the amount of interest earned is €1836 which is the same as in part (a) to the nearest euro.

Example 4:Leaving Cert Ord level 2011 Sample question 2;
(a)A sum of €5000 is invested in an 8 year government bond with an annual equivalent rate (AER) of 6% .Find the value of the investment when it matures in 8 years time.

Solution : Use the Compound Interest formula $A = P(1+\frac{R}{100})^n$.

P = €5000,R = 6% ,n = 8% . $A = €5000(1+\frac{6}{100})^8 = €7969.24$

(b) A different investment bond gives 20% interest in 8 years .Calculate the AER. .
We can again use the compound interest formula.

$$P(1+\frac{R}{100})^8 = P(1.20) \Rightarrow (1+\frac{R}{100})^8 = 1.20 \Rightarrow (1+\frac{R}{100}) = (1.20)^{\frac{1}{8}} = 1.02305 \Rightarrow$$

$$1+\frac{R}{100} = 1.02305 \Rightarrow \frac{R}{100} = .02305 \Rightarrow R = (100)(0.02305) \Rightarrow R = 2.305\%$$

Example 5;A sum of money P which was invested at 10% interest will become €159,440.49 in 6 years find P.

$$€159,440.49 = P(1+\frac{10}{100})^6 \Rightarrow P(1.771561) = €159,440.49 \Rightarrow P = \frac{€158,440.49}{1.771561} = €90,000$$

Depreciation

Depreciation this is where an asset looses value at a particular % rate over a given time .

The depreciation formula is $A = P(1-\frac{R}{100})^n$.

Example 6 ; A €30,000 car depreciates @ 10% per annum find its value at the end of 5 years .

Solution $A = €30000(1-\frac{10}{100})^5 = €17714.7$

Words

(associated with loans or other forms of credit)

Annual percentage rate (APR). The APR is the annual interest rate expressed as a % to at least 1 place of decimals that makes the present value of all these repayments equal to the present value of the loan

Words (associated with savings)

Annual Equivalent rate (AER).
Example show that an Interest rate that pays 2.75% after 9 months is equivalent to an AER of 3.68%

Solution $(1.+\frac{R}{100})^{.75}* = (1.0275) \Rightarrow 1+\frac{R}{100} = (1.0275)^{\frac{1}{.75}} = 1.0368 \Rightarrow R = 3.68\%$ (* ¾ 0f a year)

Example 7:If a Bank bond gives 50% gross after 10 years find the AER

Using the compound interest formula $A = P(1+\frac{R}{100})^n$ .We know that

$$1(1+\frac{R}{100})^{10} = 1.5 \Rightarrow (1+\frac{R}{100}) = (1.5)^{\frac{1}{10}} = 1.041379 = 1+\frac{R}{100} \Rightarrow R = 4.138\%$$

Present worth :

The is the value **now** of a payment or investment made when the interest rate is R% in n years time.

We know from the compound interest formula that $A = P(1+\frac{R}{100})^n$ Therefore the value now of P in n years time at R% is $\frac{A}{(1+\frac{R}{100})^n} = P$.

Example The Present value (value now) of €100 invested for one year at 10% is $\frac{100}{(1.1)^1}$

Example :The value now of €100 which will be received in 2 years time when the interest rate is 10% is $\frac{€100}{(1.1)^2} = €82.64$.

Geometric Series ; $a + ar + ar^2 + ar^3 ....ar^{n-1} = \frac{a(r^n - 1)}{r-1}$ a = first term r is the common ratio .

We can use Present worth and the sum of a geometric series to calculate the value of equal instalments to pay off a loan .

Example 8;

A person borrows €12000 at an APR of 4%. He wants to repay the loan in 5 equal instalments over 5 years .The first repayment is at the end of the first year after the loan has been drawn down.

The present value of the first Instalment is $\frac{A}{1.04}$, the second instalment is $\frac{A}{(1.04)^2}$

The sum of the 5 instalments is $\frac{A}{1.04} + \frac{A}{(1.04)^2} + \frac{A}{(1.04)^3} + \frac{A}{(1.04)^4} + \frac{A}{(1.04)^5} = €12000$

this is a geometric series the sum of the 5 terms of this series is €12000.

The first term is $\frac{A}{1.04}$ and the common ratio (r) is $\frac{1}{1.04}$

$$S_5 = \frac{a(1-r^5)}{1-r} = \frac{A}{1.04}\frac{(1-(\frac{1}{1.04})^5}{(1-\frac{1}{1.04})} = 4.45182(A) = €12000 \Rightarrow A = €2695.53$$

There is a quicker way to do this type of problem using the amortisation formula .

$A = \frac{PR(1+R)^n}{(1+R)^n - 1}$ .A is the equal instalment .P is the present worth of money borrowed in n years time, n the number of years/instalments .R is the interest rate .

Using the formula we get $\frac{€12000(.04)(1.04)^5}{(1.04)^5 - 1} = €2695.53$

The Amortisation formula

This is a very useful formula and can be used in many different situations such as paying off a loan in equal instalments (the formula pays the interest and principle over the period).Or saving monthly for a pension.

<u>We are required in the syllabus to derive this formula</u> .

The formula $A = \frac{PR(1+R)^n}{(1+R)^n - 1}$. A is the monthly payment paid <u>at the end of the first</u> month,R is the % interest rate P is the sum borrowed.

A person borrows €P which he intends to pay back in 5 equal instalments of € x where the interest rate is R%

At the end of year 1 he owes $P(1+\frac{R}{100}) - x$.

At the end of year 2 he owes $P(1+\frac{R}{100})^2 - x(1+\frac{R}{100}) - x$

At the end of year 3 he owes $P(1+\frac{R}{100})^3 - x(1+\frac{1}{100})^2 - x(1+\frac{R}{100}) - x$ Therefore at end of year 5 he owes

$P(1+\frac{R}{100})^5 - x(1+\frac{1}{100})^4 - x(1+\frac{R}{100})^3 - x(1+\frac{1}{100})^2 - x(1+\frac{1}{100}) - x = 0$ * pays off the loan

$\Rightarrow P(1+\frac{R}{100})^5 = x(1+\frac{1}{100})^4 + x(1+\frac{R}{100})^3 + x(1+\frac{1}{100})^2 + x(1+\frac{1}{100}) + x..$

The right hand side is a geometric series with 5 terms ,first term x common ratio $(1+\frac{1}{R})$.

$\Rightarrow P(1+\frac{R}{100})^5 = x(\frac{(1+\frac{R}{100})^5 - 1}{(1+\frac{R}{100}) - 1} .. \Rightarrow x = \frac{P(1+\frac{R}{100})^5(\frac{R}{100})}{(1+\frac{R}{100})^5 - 1}$ to find the general formula replace 5 by n .

Note the formula for repayments <u>at the start of each month</u> is $A = \frac{PR(1+R)^{n-1}}{(1+R)^n - 1}$ as

At the end of year 1 he will owe $(P-x)(1+\frac{R}{100})$ **<u>At the end of year 2 he owes</u>**

$(P-x)(1+\frac{R}{100})^2 - x(1+\frac{R}{100})$ using the method above we get $A = \frac{PR(1+R)^{n-1}}{(1+R)^n - 1}$

Example 9:

2011 question 6 Pre –Leaving Paper 1.

A graduate is setting up his own computer company he borrows €5,000 for set up costs for 6 months at 1% per month (compounded monthly).He wants to pay off in equal monthly instalments.

(i)Calculate the monthly repayments .

We use the amortisation formula $A = \dfrac{PR(1+R)^n}{(1+R)^n - 1}$ (note R = Interest rate written as a %)

P= €5000,R=1%,n= 6 $A = \dfrac{€5000\frac{1}{100}(1+\frac{1}{100})^6}{((1+\frac{1}{100})^6 - 1)} = €862.74$

Calculating Monthly Interest rates

Example If the AER is 5% find the monthly rate of interest

$$(1+\frac{R}{100})^{12} = 1.05 \Rightarrow (1+\frac{R}{100}) = (1.05)^{\frac{1}{12}} \Rightarrow 1+\frac{R}{100} = 1.004074124 \Rightarrow R = 0.4074124\%$$

Verify If the monthly interest rate is 0.4074124% find the AER . We will show that €100 becomes €105 in 12 months at this rate using the compound interest formula

$A = P(1+\frac{R}{100})^n \Rightarrow 100(1.004074124)^{12} = €105$ therefore the AER is 5% .

Question 6 LCH Sample paper

(a)Write down the present value of a future payment of €20,000 in 1 years time where the interest rate is 3%

Answer $\dfrac{€20000}{1.03}$.

(b)Write down in terms of t the present value of a future payment of €20,000 in t years time when the interest rate is 3%

Answer $\dfrac{€20000}{(1.03)^t}$

(c)Padraig wants to have a fund that from the date of his retirement will give him a payment of €20,000 at the start of each year for 25 years .

The conventional method to do this question is the add up the present values of all the instalments as follows

$$€20000 + \frac{€20000}{(1.03)^1} + \frac{€20000}{(1.03)^2} + ... + \frac{€20000}{(1.03)^{24}} = x$$

$\rightarrow €20000(1.03)^{24} + €20000(1.03)^{23} + ... €20000$ (geometric series) $= x(1.03)^{24}$

We find the sum of the series using $S_n = a(\frac{R^n - 1}{R-1}), a = €20,000, R = (1.03), n = 25$

$$20000(\frac{(1.03)^{25} - 1)}{1.03 - 1} = x(1.03)^{24} \rightarrow x = €358,710.84$$

Or we can use the **Amortisation formula as we want to find what fund is required to make a payment of €20,000 for 25 years @3%**

$A = \frac{PR(1+R)^{n-1}}{(1+R)^n - 1}$ A is the monthly payment made at the start of the month , P is the fund R is the rate rewriting the formula we get

$$\frac{A((1+R)^n - 1)}{R(1+R)^{n-1}} = P \Rightarrow \frac{€20{,}000((1.03)^{25} - 1)}{(.03)(1.03)^{24}} = €358{,}710.84$$

The Amortisation formula is used to pay off a loan and it is based on the fact that the initial payment is $A(1+\frac{R}{100})^1$ whereas our first payment is A therefore we must change n from 25 to 24 in the top line of the formula.

Example 10;

A man borrows €60,000 @ 6% AER .he wishes to pay off the loan by equal monthly instalments over 5 years.

First we need to find the monthly interest rate which is equivalent rate of 6% AER

$$(1+\frac{R}{100})^{12} = (1+\frac{6}{100})^{12} = (1.06)^{\frac{1}{12}} \Rightarrow (1+\frac{R}{100}) = (1.06)^{\frac{1}{12}} = 1.00486755 \Rightarrow R = .486755\%$$

Now use the amortisation formula $A = \frac{PR(1+R)^n}{(1+R)^n - 1}$

$$A = \frac{60{,}000(.00486755)(1.00486755)^{60}}{(1.00486755)^{60} - 1} = €1155.54$$

Example 11

A person saves €100 per month when the AER is 4% what will this amount to in 5 years .

First find the monthly interest rate $= (1.04)^{\frac{1}{12}} = 1.00327374 \Rightarrow R = .327374\%$

We can do question in 2 ways .

The monthly payments are

$100 + 100(1+\frac{R}{100})^1 + 100(1+\frac{R}{100})^2 ....+100(1+\frac{R}{100})^{59}$ This is a geometric series with 60 terms We now find the sum of the 60 terms using the formula $S_n = a(\frac{r^n - 1}{r-1})$

A= 100, common ratio $(1+\frac{R}{100})$ $S_{60} = 100(\frac{(1.00327374)^{60} - 1}{.00327374}) = 6617.90$

Method 2:

Or we could use a re-jigged form of the amortisation formula $A = \frac{PR(1+R)^n}{(1+R)^n - 1}$

We know A = €100, R = .00327374 n= 60 we wish to find P

Therefore we know that $€100 = \frac{\frac{P}{(1+R)^n} R(1+R)^n}{(1+R)^n - 1}$ Where $\frac{P}{(1+R)^n}$ is the present value of P saved in n years time when the interest rate is R%.

$$€100 = \frac{\frac{P}{(1+R)^n} R(1+R)^n}{(1+R)^n - 1} \rightarrow €100 = \frac{P(.00327374)}{(1.00327374)^{60} - 1} \rightarrow P = €6617.90$$

Sinking Fund

Sinking fund this is a sum of money that you have decided you need in the future and now you require to "sink "regular amounts to reach this target .

Example 12

Jim needs €10,000 in 3 years .How much should Jim deposit at the end of each month in an account that pays 7.5% (AER) to achieve €10,000 in 36 months.

First find the monthly rate that corresponds to AER of 7.5%

Which is $(1.075)^{\frac{1}{12}} = 1.006044919 = 0.6044919\%$ Note R = % rate /100!

We then can use the amortisation formula $A = \frac{PR(1+R)^n}{(1+R)^n - 1}, P = \frac{€10,000}{(1+R)^n}$* therefore the formula simplifies to $A = \frac{€10000(0.006044919)}{(1.006044919)^{36} - 1} = €249.48$

*The present value of €10,000 received in 3 years time when the interest rate is 7.5%

Lottery

Example 13

John wins a lottery which entitles him to €2000 at the end of each month for the next 10 years if the AER is 8% find the lump sum value of the Prize.

We need first to find the monthly Interest rate $(1.08)^{\frac{1}{12}} = 1.006434 = 0.6434\%$
The sum of the present values of the prizes is

$€2000 + \frac{€2000}{1.006434} + \frac{€2000}{(1.006434)^2} + ...\frac{€2000}{(1.006434)^{119}}$ this is a geometric series with 120 terms we use the formula $S_n = \frac{a(r^n - 1)}{r - 1}$ Take out €2000 from each term to get a = 1 and the common ratio (r)= $\frac{1}{1.006434} = (\frac{1}{(1.08)})^{\frac{1}{12}}$ therefore

$$S_{120} = \frac{€2000((\frac{1}{1.08})^{10} ** - 1)}{(\frac{1}{1.08})^{\frac{1}{12}} - 1} = €167938.39$$ **<u>This is the Lump sum value of the prize.</u>**

**Note $(\frac{1}{1.006434})^{120} = ((\frac{1}{1.08})^{\frac{1}{12}})^{120} = (\frac{1}{1.08})^{10}$

We could have also used the Amortisation formula as we are really saving €2000 per month for 10 years at an AER of 8% .
Given

$$A = \frac{PR(1+R)^n}{(1+R)^n - 1} \Rightarrow A((\frac{(1+R)^n - 1)}{R(1+R)^{n-1}} = P \Rightarrow P = \frac{€2000((1.08)^{10} - 1)}{(1.08)^{10}((1.08)^{\frac{119}{120}} - 1)} = €166864.78$$

Financial Maths

Geometric series $a + ar + ar^2 + ar^3 ....ar^{n-1}$.

The sum of the first n terms of the geometric series $S_n = \frac{a(r^n - 1)}{r-1}, |r| > 1$ $S_n = \frac{a(1-r^n)}{1-r}, |r| < 1$

Amortisation formula for calculating the **equal repayments of A** off a loan **P** in n years at i % .

The derivation of the formula depends on the statement below.

The" **sum of the present values of the payments equals the present value of the loan"**

Where the first payment is made at the end of the first payment interval (usually at end of the first month after the loan have been drawn down)

$$\frac{A}{(1+i)} + \frac{A}{(1+i)^2} + \frac{A}{(1+i)^3} + ... + \frac{A}{(1+i)^n} = P$$ (Multiply both sides by $(1+i)^n$)

$A + A(1+i) + A(1+i)^2 + ...A(1+i)^n = P(i+i)^n$ .The LHS is a geometric series

$T_1 = A, r = (1+i)$ Find $S_n = \frac{a(r^n - 1)}{r-1}, |r| > 1$

$$= A\frac{((1+i)^n - 1)}{(1+i)-1} = P(1+i)^n \rightarrow A = \frac{P(1+i)^n(i)}{(1+i)^n - 1}$$

Example :1:
Fred Borrows €10,000 with an APR of 6% he wants to repay it in 5 equal instalments the first instalment at the end of the first year .Find the amount of each repayment.

P= €10,000 ,i = 0.06, n = 5 .

$$A = \frac{€10000(1.06)^5(.06)}{(1.06)^5 - 1} = \frac{€802.94}{0.3382256} = €2379.98.$$

If Fred repays the loan in 60 monthly payments .We will need to find the monthly equivalent interest rate $= (1.06)^{\frac{1}{12}} = 1.004867551 \rightarrow i = 0.004867551$

$$A = \frac{€10000(1.004867551)^{60} **(0.004867551)}{(1.004867551)^{60} - 1}$$ it is easier to write like this

$$A = \frac{€10000(1.06)^5(.004867551)}{(1.06)^5 - 1} = \frac{€65.14}{0.3382257} = €192.59$$

** $(1.004867551)^{60} = (1.06)^5$ {This reduces the amount of "**Messy Data**")

The Amortisation Formula and <u>saving a lump sum or funding a sinking fund</u> .
This is where we save a lump sum by equal instalments P.

The Sum of the present value of the saving of all instalments A is equal to the present value of the lump sum P.

This gives $\frac{A}{(1+i)}+\frac{A}{(1+i)^2}+\frac{A}{(1+i)^3}+\ldots\ldots\ldots\ldots+\frac{A}{(1+i)^n}=\frac{P}{(1+i)^n}$. (Multiply by $(1+i)^n$).

$A+A(1+i)+A(1+i)^2+\ldots\ldots\ldots A(1+i)^n=P\frac{(1+i)^n}{(1+i)^n}=P$. Find S_n of the Geometric series

$A\frac{((1+i)^n-1)}{(1+i)-1}=P\rightarrow A=\frac{P(i)}{(1+i)^n-1}$. (Not in tables)

This is the formula where the first payment is paid at the **end of the first period.**

Example 2:

A company need to have a fund of €5000 to be available in 3 years when yearly interest rate is 8% .It intends to make 36 monthly instalments at the end of each month.

First find the monthly equivalent interest rate $(1.08)^{\frac{1}{12}}=(1.0064340300\rightarrow i=00643403$.

$$A\frac{((1+i)^n-1)}{(1+i)-1}=P\rightarrow A=\frac{P(i)}{(1+i)^n-1}=\frac{5000(.00643403)}{(1.08)^3-1}=\frac{32.17015}{0.259712}=€123.87$$

<u>Example 3:</u>

<u>Saving a set amount monthly to find what it becomes in n years @ i %</u>

We know that the following formula will tells us what we need **<u>to save per month</u>** to accumulate

$A=\frac{P(i)}{(1+i)^n-1}$ if we rearrange the formula $\frac{A((1+i)^n-1)}{i}=p$.

If I save €100 per month what will I have saved in 5 years @4%.

First find the monthly interest rate $(1.04)^{\frac{1}{12}}=1.00327374$

$$\frac{A((1+i)^n-1)}{i}=p\rightarrow\frac{100[(1.04)^5-1]}{0.00327374}=\frac{21.66529024}{0.00327374}=€6617.90$$

This is the amount saved starting at the end of the first month.

If we started saving at the beginning of the first month (far more likely) we must multiply our answer by $1.00327374=6617.90x1.00327374=€6639.57$

Financial Maths   Ballinteer Institute 01-2960006

Getting value from the amortisation formula $A = \frac{P(0R)(1+.0R)^n}{(1+.0R)^n - 1}$ .This is the formula which is used to calculate the value of the regular equal payments (€A) required to pay off a loan of €P in n instalments when the interest rate is R%

Example 1;A loan of €10,000 is repaid in 10 equal instalments of A at the end of each year for 10 years the AER is 4% find A.
Here P = €10,000 $n = 10$

$$A = \frac{P(0R)(1+.0R)^n}{(1+.0R)^n - 1} = A = \frac{€10,000(04)(1+.04)^{10}}{(1+.04)^{10} - 1} = €1232.91$$

Amortisation schedule showing the first 5 years of payments

Year	Payment (A)	Interest (B)	Payment .towards Debt (C)	Balance D
1	€1232.91	€400	€832.91	€9167.09
2	€1232.91	€366.68	€866.23	€8300.86
3	€1232.91	€332.3	€900.87	€7399.99
4	€1232.91	€295.99	€934.92	€6465.08
5	€1232.91	€258.60	€974.31	€5490.77

B = D x (0.04), C + A –B

Example 2; A person wants to save monthly for a car which they know will cost €20,000 in 3 years time .How much will they save per month if the AER is 5%.
This question is similiar to question 1.We want to find the value of the equal instalments required to accumulate €20,000 in 3 years time .We can deal with this by regarding P in the formula as the present value of €20000 payable in 3 years (36 months) time when the AER is 5% .
Since we are saving monthly we must first find the monthly % rate which is equivalent to a 5% AER $(1.0R) = (1.05)^{\frac{1}{12}} = 1.004074124 \rightarrow .0R = .004074 \rightarrow R = 0.4074\%$.

The present value of €20,000 payable in 3 years at 5% is $\frac{€20,000}{(1.05)^3} = \frac{€20000}{(1.004074)^{36}}$.

Now use the formula $A = \frac{P(0R)(1+.0R)^n}{(1+.0R)^n - 1} = A = \frac{\frac{€20000}{(1.004074)^{36}}(0.004074)(1.004074)^{36}}{(1+.004074)^{36} - 1}$

This simplifies to $A = \frac{€20000(.004074)}{(1+.004074)^{36} - 1} = €294.92$

Example 3:
A variation of the formula can be used to find the sum regular savings
A person saves €100 per month when the AER is 4% what will this amount to in 5 years

First find the monthly interest rate $=(1.04)^{\frac{1}{12}}=1.00327374 \Rightarrow R=.327374\%$.
The 1st deposit of €100 becomes $100(1.00327374)^{60}$ in 5 years the last €100 becomes $100(1.00327374)^{1}$ The total monthly payments are
$100(1.00327374)+100(1.00327474)^{1}+100(1.00327374)^{2}..........+100(1.00327374)^{60}$ This is a geometric series with 60 terms We now find the sum of the 60 terms using the formula $S_n=a(\frac{r^n-1}{r-1})$ Take out the common factor €100 this gives the geometric series
$€100\{(1.00327374)+((1.00327374)^2+.......+(1.00327374)^{60}$

$$S_{60}=€100(1.00327374)(\frac{(1.00327374)^{60}-1}{.00327374}=€6639.57$$

Looking at this result we see that the sum that $P=\frac{A(1.0R)\{(1.0R)^n-1)}{.0R}$
Where A = €100,P = €6639.57 .In this case we know both A and P but if we just knew A, R% and n .

We could rewrite the formula as $\frac{P(.0R)}{((1.0R)^n-1)}\frac{1}{1.0R}=A$ This is the amortisation formula from above in example 2 divided by (1.0R) .You can use this anytime you are trying to find the sum of regular savings .

Example 4
€100 is invested at the beginning each year for 5 consecutive years .If the interest rate is 10% find the value of the investment at the end of the 5th year .
From above A = €100 .0R =.1, n= 5
Using $\frac{P(.0R)}{((1.0R)^n-1)}\frac{1}{1.0R}=A\ \frac{P(.1)}{((1.1)^5-1)}\frac{1}{1.1}=100 \rightarrow 0.1489068P=100 \rightarrow p=€671.56$

Or Just use $P=\frac{A(1.0R)\{(1.0R)^n-1)}{.0R}$

Main points(i)Amortisation Formula used to calculate the equal payments on a loan

$$A=\frac{P(0R)(1+.0R)^n}{(1+.0R)^n-1}$$

(ii)Amortisation Formula used to find the fixed repayment to fund a future amount

$$A=\frac{P(.0R)}{(1+.0R)^n-1}$$

The Amortisation Formula three uses.

If **P** is the sum of money borrowed If **I** is the interest rate n is the number of repayments and A is the value of the monthly/yearly repayments

$$A = \frac{P(1+i)^n(i)}{((1+i)^n - 1)}$$

Example 1; €20000 is borrowed and paid off in equal monthly repayments over 5 years when

APR is 5%. If repayments are monthly we must find the monthly interest rate

$(1.05)^{\frac{1}{12}} = (1+i) \rightarrow (1.004074124) \rightarrow i = 0.004074124$ monthly interest rate is 0.40742124%.

Find the value of the monthly repayment.

$$A = \frac{20000(1.05)^5(.004074124)}{((1+05)^5 - 1)} = €376.41$$

Example 2: A person wishes **to save €20,000** in 5 years by paying a certain amount at the end of every month when the APR is 5% .Find the monthly amount.
We can use the amortisation formula **where P is the present value of €20000** payable in 5 years @APR 5%

$$A = \frac{\frac{P}{(1+i)^n}(1+i)^n(i)}{((1+i)^n - 1)} = \frac{P(i)}{((1+i)^n - 1)} \frac{€20000(.004074124)}{((1.05)^5 - 1)} = €294.93$$

Example 3: A persons saves €100 per month when APR is 5% how much will he save in 5 years?

We know from example 2 that $A = \frac{P(i)}{((1+i)^n - 1)} \rightarrow \frac{A((1+i)^n - 1)}{(i)} = P$ we can use this form of the formula to find P.

$$\frac{€100((1+.05)^5 - 1)}{(.004074124)} = P = €6781.37$$

Note $(1.004074124)^{60} = (1.05)^5$ this can help solving rounding up problems.