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Leaving H Maths

Mathematical Induction

Complex Numbers

Functions and Calculus

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Notes on Induction

Induction is a process which uses Three steps to show that a given statement is true

ex we can show by induction that $\sum n^2 = \frac{n(2n+1)(n+1)}{6}, n \in N_0$,, or that $7^{2n+1}7^{2n+1} + 1, n \in N \geq 1$ is divisible by 8, or$(1+x)^n \geq 1+nx, x > -1, n \in N_0$. Note all the statements refer to n as a positive integer , Induction is a process which applies to statements involving **positive integers** only !.

The process works as follows (1) We show that the statement is true for **n = 1** , (2) We assume that it is true for **n= k** , (3) We prove the statment is true for **n = k+1** . The idea being that if the statement is true for n = 1 and is true for n = k +1 , if true for n = k . Then if n = 1 it's true for 1+1, if true for 2 then it's true for 2 + 1 , etc for all positive integers .

One way in which Induction is described is that of an Infinite ladder and the rungs of the ladder are so spaced that if you are on any rung you know you can get to the next rung, which means if you can get to the first rung you can climb the ladder as high as you want ,

Mathematical Induction is used in the following situations

(1) To show that a given expression in n represents the sum of the first n terms of a particular Series .

(2) To show that certain Inequalities are true .

(3)To Show that a given expression is divisible by a given number .

(4) To show DeMoivres theorem is true,

Sums of Series

Example 1 Prove $\sum_1^n r^2 = 1^2 + 2^2 + 3^2 + 4^2 ....n^2 .= \frac{n(2n+1)(n+1)}{6} = S_n$.

(1)For n = 1. We show $S_1 = 1 = \frac{1.3.2}{6} = 1.$ the statement is true .

(2)We now assume that the statement is true for n = k , ie $S_k = \frac{k(2k+1)(k+1)}{6}$

(3) We now prove that the statement is true for n = k+1. ie $S_{k+1} = \frac{(k+1)(2k+3)(k+2)}{6}$.

The way to prove this is follows $S_{k+1} = S_k + U_{k+1} =$

$$\frac{k(2k+1)(k+1)}{6} + (k+1)^2 ..= \frac{k(2k+1)(k+1)+6(k+1)^2}{6}$$

$$= \frac{(k+1)\{k(2k+1)+6(k+1)\}}{6} = \frac{(k+1)\{2k^2+7k+6)\}}{6} = \frac{(k+1)(2k+3)(k+2)}{6} = S_{k+1}$$

We have shown the statement to be true for n = 1, and if true for n = k , it's true for n = k + 1 ,

This Method of using $S_{k+1} = S_k + U_{k+1}$ will always work for sums of series .

Factorisation Results ("is divisible by")

The srandard method here is to write the expression for k + 1 in terms of the expression in terms of k and a remainder , the remainder must be divisible by the given number .

Ex (2) Prove $5^n - 3^n$ is divisible by 2 .If n = 1 we get 5 - 3 = 2 which is divisible by 2 .

Assume the statement is true for n = k , ie $5^k - 3^k$ is divisible by 2 .

Now prove the statement is true for n = k+ 1 . ie $5^{k+1} - 3^{k+1}$ is divisible by 2

$.5^{k+1} - 3^{k+1} = 55^k - 33^k = 3(5^k - 3^k) + (2)5^k$

We know$5^k - 3^k$ is divisible by 2 and $2(5^k)$ is divisible by 2 therefore the statement is true for n = k+1

Inequalities

Prove $2^n > n^2, .. n > 4$

(1)Show the statement is true for n = 5 . 32 .> 25 True we use 5 as n > 4 .

(2)Assume the statement is true for n = k , $2^k > k^2$.

(3) Prove true for n = k + 1 . $2^{k+1} > (k+1)^2 \Rightarrow 2(2^k) > k^2 + 2k + 1, .. \Rightarrow$
$2^k + 2^k > k^2 + 2k + 1, but. 2^k > k^2 .. and. 2^k > 2k + 1 .. for .. k > 4.$

Note in step (3) the RHS is a product wheras the LHS is a Sum it is important to <u>change both sides</u> into a SUM or a product .

Example 2:

Prove $2^n \geq 1 + n, \forall . n \in N$

(1)Prove true for n = 1. $2 \geq 1 + 1 .. \Rightarrow 2 \geq 2 .. true! \rightarrow$

(2)Assume true for n = k . $2^k \geq k + 1$. Now prove true for n = k+1

$$2^{k+1} \geq k + 1 + 1 \Rightarrow 2(2^k) \geq k + 1 + 1 \Rightarrow 2^k + 2^k \geq k + 1 + 1$$

$$\textbf{but}..2^k \geq k + 1 .. \textbf{and}. 2^k \geq 1, .. \textbf{for} .. k \in N.$$

Example 3.

Prove $(1 + x)^n \geq 1 + nx, .. x > 0$.

(1)Prove true for n = 1. $(1 + x) \geq 1 + x .. \text{true!}$

(2)Assume true for n = k . ie $(1 + x)^k \geq 1 + nx$. now prove true for n = k+1 .

(3) $(1 + x)^{k+1} \geq 1 + (k+1)x \Rightarrow (1 + x)^k (1 + x) \geq 1 + kx + x \Rightarrow$
$(1 + x)^k + x(1 + x)^k \geq (1 + kx) + x .. \text{but} (1 + x)^k \geq (1 + kx) .. \text{and} .. x(1 + x)^k \geq x, .. \forall x > 0$

therefore the statement is true for n = k + 1 .

Some interesting Examples :

Show $x^n - y^n$ is divisible by x - y . (1) Show true for n = 1 , ie (x - y) is divisible by (x - y)

(2) Assume true for n = k . ie $x^n - y^n$ is divisible by x - y .

(3) Prove true for n = k + 1 . is $x^{k+1} - y^{k+1} = x^{k+1} - x^k y + x^k y - y^{k+1} = x^k (x - y) + y(x^k - y^k)$

To prove the statement true we add and subtract $x^k y$ we can see from the result that (x - y) is a factor of both parts .

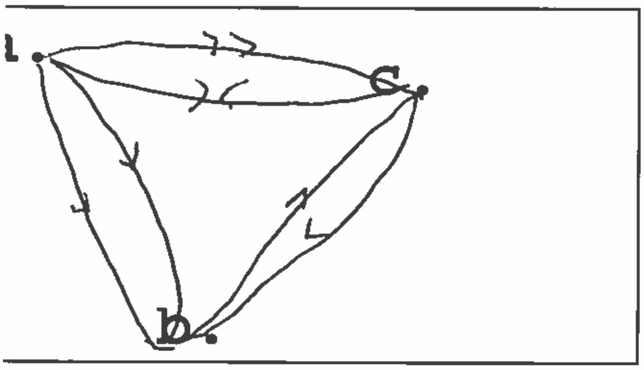
Prove that (n)(n-1) couples (excluding identity couples) can be formed from n points .

(1)When n = 2 we get two couples =2(2-1) = 2 true for n = 2 .

(2)Assume true for **n = k ie we get k(k-1) couples from k points**

(3) Prove true for **n = k + 1 . ie we should get (k+1)k couples from k + 1 points** .

For **n = k we get k(k-1)** couples when n = k + 1 (**we get an extra 2k couples so the total number of couples from k + 1 points is k(k - 1) + 2k = k(k + 1) the statement is true for n = k + 1 .



** if you have k points and add an extra point yuo will get an extra k couples

When we have two points a and b we have two couples (a,b) and (b,a) When we add the point c we get 4 more couples ie (a,c),(c,a),(c,b)(b,c) that is 2 couples for each point that is already there .

Complex Numbers **Ballinteer Institute,** **Leaving Cert maths**

Solve $x^2 = 25 \Rightarrow x = \pm 5$ what happens if we are asked to Solve $x^2 = -25 \Rightarrow x = \pm\sqrt{-25}$ now if you try to find the $\sqrt{-25}$ using your calculator you will get" errror 1 or error 2 "as the $\sqrt{-25}$ is not a real number to cope with this we use the following technique $\sqrt{-25} = \sqrt{25(-1)} = \sqrt{25}\sqrt{-1} = 5i$, **i** represents the $\sqrt{-1}$.

Since **i** represents the $\sqrt{-1} \Rightarrow i^2 = -1 \Rightarrow i^3 = i(i)^2 = i(-1) = -i \Rightarrow i^4 = (i^2)^2 = (-1)^2 = 1$

Numbers of the form **a+ib** are called **Complex Numbers** where a and b are Real Numbers and i is $\sqrt{-1}$.

Algebra of Complex Numbers

(1) **Addition** $(a+ib)+(c+id) = a+c+i(b+d)$ Example $3+4i+5+2i = 8+6i$

(2)**Subtraction** $(a+ib)-(c+id) = a-c+i(b-d)$ Example
$4+7i-(3+2i) = 4-3+(7-2)i = 1+5i$

(3)**Multiplication of a Complex Number by a Real number** Ex : 3(4+5i)= 12 + 15i

(4) **Multiplication of a Complex Number by a Complex Number** Must remember that $i^2 = -1$
Ex : $2i(3+4i) = 6i+8i^2 = 6i-8 = -8+6i$

Ex: $(2+3i)(4-5i) = 8-10i+12i-15i^2 = 8+2i+15 = 23+2i$

Complex Conjugate Every Complex number **a + i b** has a conjugate **a-ib** .
if $z = a+ib, .. \bar{z} = a-ib$

When we multiply a Complex Number by its conjugate we get the following result
(a+ib)(a-ib)$= a^2 - abi + abi - (ib)^2 = a^2 - (-1b^2) = a^2 + b^2$ a **real number** note the result is $a^2 + b^2$ (no i's) are involved :

Example ;$(3+4i)(3-4i) = 3^2 + 4^2 = 25 ... or (-1-i)(-1+i) = (-1)^2 + (1)^2 = 2$

Division in Complex Numbers Example
(1) Division of a Complex Number by a real number
$(6+15i) \div 3 = 2+5i$

((2) **Division of a Complex Number by a Complex Number**
(multiply above and below by the Conjugate of the bottom line)

$$(2+3i) \div (3-4i) = \frac{2+3i}{3-4i} \times \frac{3+4i}{3+4i} = \frac{6+8i+9i-12}{3^2+4^2} = \frac{-6+17i}{25} = \frac{-6}{25} + \frac{17i}{25}$$

Complex Number Equations :

Set the **Reals** = to the **Reals** and the **i's** = to the **i 's** .

Example (1) Solve for a and b
$3+4i+7+5i=a+bi:\Rightarrow 3+7=a,\Rightarrow a=10,\Rightarrow b,=4+5, a=10, b=9.$

Example (2) Solve for s and t

$s(2-i)+ti(4+2i)=1+s+ti \Rightarrow 2s-is+4ti-2t=1+s+ti \Rightarrow$

$2s-2t=1+s,\Rightarrow -s+4t=t.\Rightarrow \begin{matrix} s-2t=1 \\ -s+3t=0 \end{matrix} \Rightarrow t=1, s=3.$

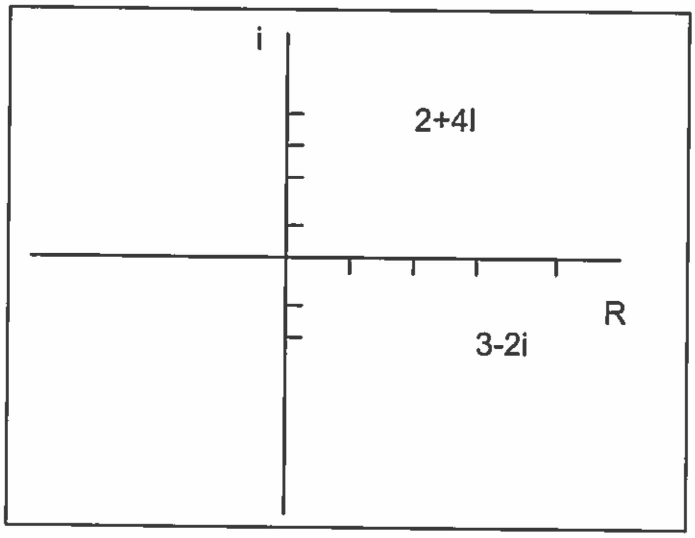
Complex Numbers and Quadratic Equations :
Solve $x^2+6x+13=0 \Rightarrow x=\frac{-6\pm\sqrt{6^2-4(1)(13)}}{2}=\frac{-6\pm\sqrt{36-52}}{2}=\frac{-6\pm\sqrt{-16}}{2}=\frac{-6\pm 4i}{2}=-3\pm 4i=3+2i,3-2i$

Some important information

If the quadratic equation $x^2+bx+c=0, a,b,\in R$ has a root $x=p+iq$ then (i) $x=p-iq$ is also a root and (ii) $p+iq+p-iq=-b \Rightarrow 2p=-b$ and (iii) $(p+iq)(p-iq)=c \Rightarrow p^2+q^2=c$

Argand Diagram : A coordinate Plane for Complex Numbers , the X axis is the Real axis the Y axis is the i axis .

Plot 3-2i and 2+4i



Modulus of a Complex number.
This is the distance from (0,0) to the complex number.
The modulus of a + ib is $|a+ib|=\sqrt{a^2+b^2},\Rightarrow |3+4i|=\sqrt{3^2+4^2}=5.$
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Complex Numbers

If $z = x + iy$ the conjugate of z written as $\bar{z} = x - iy$ z and it's conjugate have the following properties (1) $z.\bar{z} = (x+iy)(x-iy) = x^2 + y^2$.

(2)
$$z + \bar{z} = (x+iy) + (x-iy) = 2x$$
$$z - \bar{z} = (x+iy) - (x-iy) = 2iy$$

$$z_1 = x + iy, .. z_2 = a + ib$$

(3)To Show

$$\left(\overline{z_1 + z_2}\right) = \bar{z}_1 + \bar{z}_2$$

$$z_1 + z_2 = (x + iy + a + ib) = a + x + i(b+y), \overline{z_1 + z_2} = a + x - i(b+y) = x - iy + a - ib = \bar{z}_1 + \bar{z}_2$$

$$\overline{z.z} = \bar{z}.\bar{z} \Rightarrow z.z = (x+iy)(a+ib) = ax - by + i(bx+ay). \Rightarrow \overline{z\ .z} = ax - by - i(bx+ay)$$

$$\bar{z}.\bar{z} = ax - by - i(xb + ay)$$

5 From above we can deduce the following
$$\overline{z_1 + z_2 + z_3} = \overline{(z_1 + z_2)} + \bar{z}_3 = \bar{z}_1 + \bar{z}_2 + \bar{z}_3.$$
$$\overline{z_1 z_2 z_3} = \overline{(z_1 z_2)}\ \bar{z}_3 = \bar{z}_1 . \bar{z}_2 . \bar{z}_3$$

We can also deduce (3) and (4)that $n(\bar{z}) = \overline{nz}, (\bar{z})^n = \bar{z}^n$ We can use the above properties to prove the conjugate roots theorem .

Conjugates Roots theorem :

If z = p +iq is a root of the cubic equation $ax^3+bx^2+cx + d$ where a,b,c,d are all real . We want to show that $\bar{z}$ = p-iq is also a root . Since z is a root then f(z) = 0.ie

f(z)= a z^3 + bz^2 + cz + d = 0 . Take the conjugate of both sides $\overline{f(z)} = \overline{az^3 + bz^2 + cz + d} = 0 \Rightarrow a(\overline{z^3}) + b(\overline{z^2}) + c\bar{z} + d = 0 \Rightarrow a(\bar{z})^3 + b(\bar{z})^2 + c\bar{z} + d = 0 \Rightarrow f(\bar{z}) = 0$

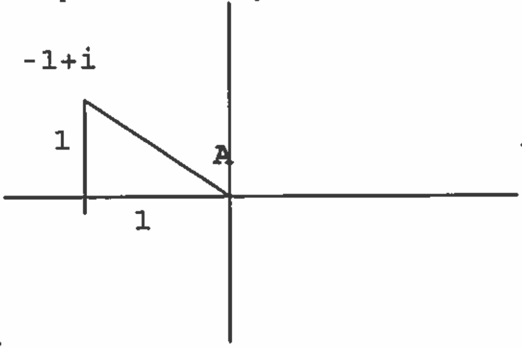
= $\bar{z}$ is also a root .

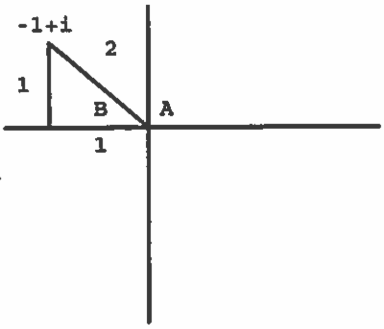
Writing Complex numbers in Polar Form

Every Complex Number x+iy can be written in the form R (Cos A + iSinA) where R =$\sqrt{x^2 + y^2}$ is called the modulus of the Complex number and A is called the argument of the Complex number. A is the angle between the line joining the Complex Number with the origin and the + direction of the X axis(similiar to slope)

A lies between -180 and 0 or between 0 and 180 .

Example write -1+ i in polar form Method (1) put -1+i on the Argand Diagran . (2) Find R and Find A





R=$\sqrt{2}$ from the diagram

we can see that *Tan B is 1 $\Rightarrow$ B = 45 which means that A is 135 (180-45)

and - 1+ i is $\sqrt{2}(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$

***Since A is greater than 90 we find B first using the triangle the take B from 180 to find A note A is always written as an angle between 0 and 180 or an angle between 0 and -180 .**

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Common Problems Solved Using Complex numbers :

The general rules of Algebra apply to Complex Numbers , with the following additions (1) $i^2 = 1$
(2) In equations involving Real Numbers and Complex numbers **Reals = Reals , i 's = i' s .**

Example 1 :u = 4 + 3i and v = a + bi , Find a and b if uv = 10 -5i , $a.b \in R$
u.v = (4 + 3i)(a + bi) 10-5 i = 4a+4ib+3ib-3b,$\Rightarrow$ $4a - 3b = 10, \Rightarrow 3a + 4b = -5$
4a- 3b = 10 ×4 16a -12b = 40 Add to get 25 a = 25 : a = 1 ,

Example 2 : Find the real numbers s and t such that $(s+it)^2 = -3+4i$

$\Rightarrow s^2 + 2sit - t^2 = -3 + 4i \Rightarrow s^2 - t^2 = -3,..2sit = 4i \Rightarrow st = 2 \Rightarrow t = \frac{2}{s}$ this gives

$s^2 - \left(\frac{2}{s}\right)^2 = -3 \Rightarrow s^4 + 3s^2 - 4 = 0 \Rightarrow (s^2+4)(s^2-1) = 0 \Rightarrow s^2 = 4,..s^2 = 1$ $s^2 = 4$ has no real solutio
and $s^2 = 1$ gives s = 1 ans s = - 1 this gives 2 values for t t= 2 , t = - 2 .

<u>**Rules which apply to Complex Numbers which are in polar form.**</u>

Rule 1 $p(CosA + iSinA)).q(CosB + iSinB) = Pq\{Cos(A+B) + iSin(A+B)\}$

Rule 2 $p(CosA + iSinA) \div q(CosB + iSinB) = \frac{p}{q}\{Cos(A-B) + iSin(A-B)\}$

Applications of above Rules Write in the form $Cos\theta + iSin\theta$

$$\frac{1+iTan\theta}{1-iTan\theta} = \frac{1+i\frac{Sin\theta}{Cos\theta}}{1-i\frac{Sin\theta}{Cos\theta}} = \frac{Cos\theta + iSin\theta}{Cos\theta - iSin\theta} = \frac{Cos\theta + iSin\theta}{Cos(-\theta) + iSin(-\theta)} = Cos2\theta + iSin2\theta$$

Applictions of De Moivre's Theorem

De Moivres Theorem States that : $(Cos\theta + iSin\theta)^n = (Cos(n\theta) + iSin(n\theta))$
Using De Moivre

Ex 1 Evaluate $\left(Cos\frac{\pi}{3} + iSin\frac{\pi}{3}\right)^5 = \left(Cos\frac{5\pi}{3} + iSin\frac{5\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

Ex 2 :Express $\frac{\sqrt{3}+i}{\sqrt{3}-i}$ in the form $Cos\theta + iSin\theta$. First write each Complex number in Polar Form .

$|(\sqrt{3}+i)| = 2,..Tan\theta = 1/\sqrt{3} \Rightarrow \theta = \pi/3... \Rightarrow \sqrt{3}+i = 2\{Cos\frac{\pi}{3} + iSin\frac{\pi}{3}\}$

$|(\sqrt{3}-i)| = 2,..Tan\theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{3}(anticlockwise) \Rightarrow \sqrt{3}-i = 2\{Cos-\frac{\pi}{3} + iSin-\frac{\pi}{3})$

Therefore $\frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{2\{Cos\frac{\pi}{3} + iSin\frac{\pi}{3}\}}{2\{Cos-\frac{\pi}{3} + iSin-\frac{\pi}{3}\}}$ (divide the moduli subtract the arguments)

$Cos\frac{2\pi}{3} + iSin\frac{2\pi}{3}$

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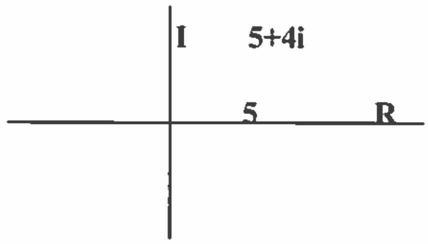
1999 Q4 Paper 1 Complex numbers



(A)
Z = 5+4i where $i^2 = -1$.
Plot (1) z, (2) z – 4i on an Argand diagram

Z = 5+ 4i, Z-4i = 5+4i-4i = 5 +0i **(10 Marks)**

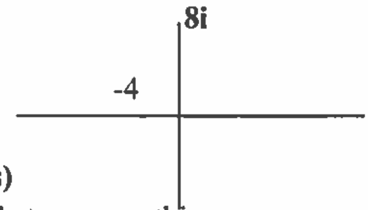
(b)(1) Let u =3 – 6i we are asked to find /3 – 6i / this the modulus of the complex number
$|x+iy| = \sqrt{x^2+y^2} \Rightarrow |3-6i| = \sqrt{3^2+(-6)^2} = \sqrt{45}$ **(10 Marks)**

(ii) **Show** $iu + \frac{u}{i} = 0$. **Multiply everything by the common denominator (i) this is the standard way to solve an equation where a bottom line is involved, this gives**
$i^2u + u = 0 \Rightarrow -1u + u = 0$ True. Another idea when you get a "solve" problem in complex numbers is just to fill in the equation with the things that you know and take it from there! Eg
$iu + \frac{u}{i} = 0 \Rightarrow i(3-6i) + \frac{3-6i}{i} = 0$, By doing this you are guaranteed the attempt mark!
You can then multiply everything by i and tidy it up. **(5 Marks)**

(iii) In this last part of part b we are asked to express $\frac{u}{u+3i}$ in the form p + q I.
Solution
First replace all the u's by 3 – 6i. This gives
$$\frac{3-6i}{3-6i+3i} = \frac{3-6i}{3-3i} = \frac{3-6i}{3-3i} \times \frac{3+3i}{3+3i} = \frac{9+9i-18i-18i^2}{9-9i+9i-9i^2} = \frac{9-9i+18}{9+9} = \frac{27-9i}{18} = \frac{3-i}{2} = 3/2 - i/2$$
(10marks)
(c) **Here we are given w = i - 2.**
Solution
We are asked to find w^2 this just means multiply w by w. this gives
$w^2 = (i-2)(i-2) = i^2 - 2i - 2i + 4 = -1 - 4i + 4 = 3 - 4i$ **(5Marks)**

In the second part we are asked to solve $kw^2 = 2w + 1 + ti$ for real k and real t.
Solution
Just replace w and w^2 in the equation. This gives k (3-4i) = 2(i – 2)+1+ ti
3k – 4ki = 2i – 4 +1 + ti. Now set reals equal to the reals and the i = i.
This gives $3k = -4+1 \Rightarrow 3k = -3 \Rightarrow k = -1$, and
$-4ki = 2i + ti \Rightarrow -4k = 2 + t \Rightarrow -4(-1) = 2 + t \Rightarrow t = 2$ **(10 marks)**
Comments:Tasks involved in this question (1)plot on an Argand diagram(a),(2)Solve a Complex Number equation(c)(3)Multiply(c)(4)Divide(b)Modulus(b)

1998 Leaving Cert Lower
Question 4 Paper 1.



(A)
Given w = 2i plot (1)..w^2...(2)..w^3 on an Argand diagram.
$w^2 = (2i)(2i) = 4i^2 = -4,..w^3 = (2i)(2i)(2i) = -4(2i) = 8i$ **(10 marks)**
Very easy same thing was asked in '99,98.'96,95 so you must be able to answer this.

(b) This part is based on the roots of a quadratic equation .We are asked to show 4 – 3i is a root of $z^2 - 8z + 25 = 0$ and find the other root, probably the easiest way is to use the roots formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow z = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(25)}}{2(1)} \Rightarrow \frac{8 \pm \sqrt{64-100}}{2} = \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm \sqrt{36}\sqrt{-1}}{2}$$

$$\frac{8 \pm 6i}{2} \Rightarrow z = 3 + 4i, z = 3 - 4i.(10.Marks)$$

(b)(ii) Investigate if $|2 + 14i| = /10(1-i)\|$ here we are asked to find the modulus of two complex numbers
Solution;
$|2 + 14i| = \sqrt{2^2 + 14^2} = \sqrt{200}$, Now find the modulus of 10-10i $= \sqrt{10^2 + (-10)^2} = \sqrt{200}$.
The results are equal! **(10 marks)**

© Given u =2 – i. We are asked to write $u + \frac{1}{u}$ in the form a +bi.

Solution
Again very easy
$2 - i + \frac{1}{2-i}$, First get $\frac{1}{2-i}$ in the form a + ib as follows

$$\frac{1}{2-i} = \frac{1}{2-i} \times \frac{2+i}{2+i} = \frac{2+i}{2+2i-2i-i^2} = \frac{2+i}{5} \Rightarrow 2 - i + \frac{1}{2-i} = 2 + i + 2/5 + i/5 = 12/5 + 6i/5$$

(10 Marks)
(ii) In the last part of this question we use the information found in part (i) to solve the equation $k(u + \frac{1}{u}) + ti = 18$,
Solution
Just fill in the parts that you know $k(12/5 + 6i/5) + ti = 18$
$k(12/5) = 18 \Rightarrow 12k = 90 \Rightarrow k = 7.5,:$
$k6i/5 + ti = 0i \Rightarrow 6k + 5t = 0 \Rightarrow 6(7.5) + 5t = 0 \Rightarrow 45 = -5t \Rightarrow t = -9$ **(10 marks)**
Comment: Tasks (1)Plot includes w^3 (there must have been lots of errors with this in'97)
(2)Modulus (b)Divide(c)(3)Complex Number equation(c)(4)Roots of a quadratic(b)

Leaving Cert ordinary Level 1997.
Question 4 paper 1.
(a)
Here we are asked to get rid of the brackets and tidy it up
Solution.
3(1 +5i)+ i (3 – 2i) = 3 +15i + 3i-2(-1) = 5+ 18 i. This was worth **10 marks!**

(B)(i) Another question based on the modulus of a Complex number
In this case we are asked to find the values of a for which $|a+8i|=10$?

Solution

$\sqrt{a^2+8^2}=10 \Rightarrow a^2+64=100 \Rightarrow a^2=36 \Rightarrow a=\pm 6$ **(10marks)**

(ii) Given w = 4i we are asked to verify $w^3-w^2+16w-16=0$
Solution just replace w in the equation by 4i

$(4i)^3-(4i)^2+16(4i)-16=0 \Rightarrow (4i)(4i)(4i)-(4i)(4i)+64i-16=0 \Rightarrow -64i+16+64i-16=0$
true **(10 Marks)**

© In this part of the question we have to divide two complex numbers and solve an equation .

We are given z =1 +i . We are asked to find $\frac{z}{\bar{z}}=\frac{1+i}{1-i}=\frac{1+i}{1-i}\times\frac{1+i}{1+i}=\frac{1+i+i+i^2}{1+i-i-i^2}=\frac{2i}{2}=i$

(10marks)

Now we are asked to use this result to Solve (very similar to '98,)
Solution :

$k\left(\frac{z}{\bar{z}}\right)+tz=-3-4i \Rightarrow ki+t(1+i)=-3-4i \Rightarrow ki+t+ti=-3-4i \Rightarrow t=-3,$ **(10 marks)**

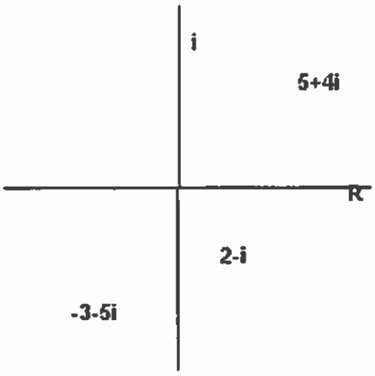
$k+t=-4 \Rightarrow -3+k=-4 \Rightarrow t=-1$

Comments This was a very short question ,You were not asked to plot on an Argand diagram here but all the rest of the usual tasks are here. (1)Multiply(a),(2)Conjugate(c) (3)complex number equation (c)(4)modulus(b).
What was a bit unusual was the second part of (b) it had not been asked before but as you can see from above it really was an exercise in multiplying out complex numbers.

1995 Leaving Cert Ordinary Level Question 4 Paper 1

(a) Given $z_1 = 5+4i,..z_2 = -3-5i,..i^2 = -1$. We are asked to plot the following $z_1, z_2, z_1 + z_2$. this is very straightforward since we know the first two numbers already

To get $z_1 + z_2$ just add 5+4i + -3-5i = 2-i for doing this you got **<u>10 marks!</u>** <u>(2 marks for axes)</u>



(b) This consists of three parts

(1) We must change w = $\dfrac{1+i}{2-2i}$ into the form p+iq .

(2) Solution: We do this by multiplying above and below by the conjugate of the bottom-line this gives

$$\frac{1+i}{2-2i} \times \frac{2+2i}{2+2i} = \frac{2+2i+2i+2i^2}{4+4i-4i-4i^2} = \frac{4i}{8} = \frac{i}{2} = W$$ **(10marks)**

(2) We are asked for the modulus of w = $\sqrt{0+\frac{1}{2}^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}$.

(5 marks)

(3) We are asked to show modulus squared is equal to w multiplied by it's conjugate i.e. is

$(\frac{1}{2})^2 = \frac{i}{2} \times \frac{-i}{2} \Rightarrow \frac{1}{4} = \frac{1}{4}$ Which is true! **<u>(5 marks)</u>**

(3) Again three parts two of which are Complex number equations.

(1) Given u = 6 – 5i find a and b if u + ai = 2b

Solution Just replace u by 6 – 5i and set reals equal to reals and i's = to i's.

This gives 6-5i + ai = 2b $\Rightarrow 6 = 2b \Rightarrow b = 3, ai = -5i \Rightarrow a = -5$ **.10 marks**

The next part is just a slightly more complicated version of the last part; again it's a Complex number equation.

(2) Solve for real s and t. s (2-i) +ti (4+2i) = 1 + s + ti

Solution just multiply it out and set reals equal to reals and i = i.

$2s - is + 4ti + 2ti^2 = 1 + s + ti \Rightarrow 2s - is + 4ti - 2t = 1 + s + ti$

$\Rightarrow 2s - 2t = 1 + s \Rightarrow s - 2t = 1 \Rightarrow s - 2t = 1$ **(5marks)**

$\Rightarrow is + 4ti = ti \Rightarrow s + 4t = t \Rightarrow s - 3t = 0 \Rightarrow t = 1, s = 3 \Rightarrow s + it = 3 + i$

A bit long but not difficult, this type of question is a regular on LCMaths paper 1.

(3) The Last part of this question was a bit off the wall it falls into the category of ask a quadratic at all costs. You are told that a complex number Z = x + iy we are asked what type of curve is represented by $|z|^2 = |s + it|^2$

Solution

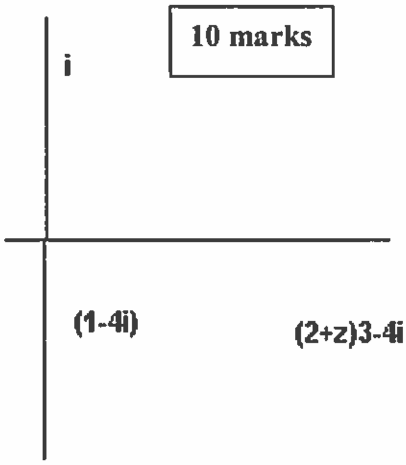
$|x+iy|^2 = |s+it|^2 \Rightarrow (\sqrt{x^2+y^2})^2 = (\sqrt{3^2+1^2})^2 = x^2 + y^2 = 10$ a circle. **(5 marks)**

The way the marks are allocated clearly shows how badly parts (2) and (3) were attempted!

Comments Last bit of part c was unusual but all of the regular tasks were asked again (1)Plot(a)(2)Divide(b)(3)Complex Number equations(c)(4)Modulus (b)(5)Conjugate(b)

1996 Leaving Cert Ordinary Level paper 1 Question 4.

(a) Given z = 1 –4i as usual we are asked to plot Complex numbers on the Argand Diagram. The want us to plot z and 2 +z Z = 1 – 4i, 2 + z = 2 + 1 – 4i=3-4i.
Solution



Part (b) Again consists of three parts this was a feature of this question up to 1996 thereafter just 2 parts.

(a) w = (1-3i)(2+i). Here we are asked to multiply out two complex numbers.

Solution

$(1-3i)(2+i) = 2+i-6i-3i^2 = 2+i-6i+3 = 5-5i = w$

(10) marks

(2) Based on the modulus and the conjugate of w.

Show $\left|w+\bar{w}\right| = \left|w-\bar{w}\right|$

Solution. Just sub in for w and it's ($\bar{w} = 5+5i$) **conjugate to get** $|5-5i+5+5i| = \|5-5i-(5+5i)\|$

$|10| = |-10i| = 10$ **True (10 marks)**

(3) We are asked to find a if $\frac{\bar{w}}{2i} = aw$. this is a Complex number equation.

Solution Just fill in for w and $\bar{w}$ and cross multiply, then set reals equal to reals and i's = to i's. This gives

$\frac{5+5i}{2i} = a(5-5i) \Rightarrow 5+5i = 2i(a)(5-5i) \Rightarrow 5+5i = 10ai - 10ai^2 \Rightarrow 5i = 10ai \Rightarrow \frac{5}{10} = a$.

(5 marks)

© Based on a Quadratic Equation:

Given z =2 – i is a root of $z^2 + pz + q = 0$ **find p and q.**

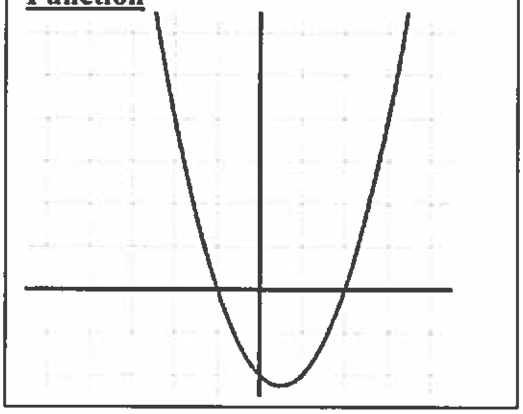
Solution: Since 2 – i is a root then 2 + i is also a root. We know the sum of the roots is –p this gives $2-i+2+i = -p \Rightarrow 4 = -p \Rightarrow p = -4$, **and the product of the roots is q this gives** $(2-i)(2+i) = q \Rightarrow 4+2i-2i-i^2 = q \Rightarrow 5 = q$ **(15 Marks)**

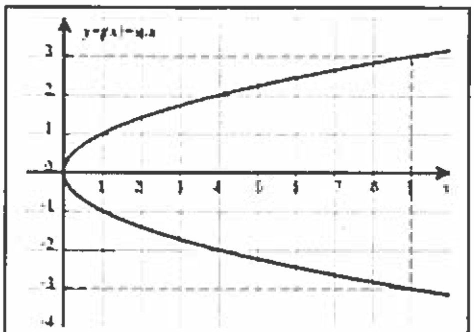
Comments Contains all of the usual tasks .
(1)Plot (a) (2)Multiply (b) (3)complex equation(b). (4)modulus and conjugate (b)(5)Roots of a Quadratic(c)

Functions Leaving Cert higher Maths

Ordered pairs or couples {(1,a),(2,b),(3,c)}
A function is a set of couples derived from a rule .It is also referred to as a mapping from one set to another .
The set of elements that can be put into the function is call the DOMAIN of the function .The set of elements that can come out of the function is called the CODOMAIN.
The set of the actual elements that come out of the function is called the RANGE of the function.
To be a function no element of the Domain can be mapped onto **more that one element** of the Codomain .
Every element of the Domain has an image in the Codomain
In general if the line $x = k$ cuts the graph of $y = f(x)$ **at most once then** $f(x)$ is a function .

Function





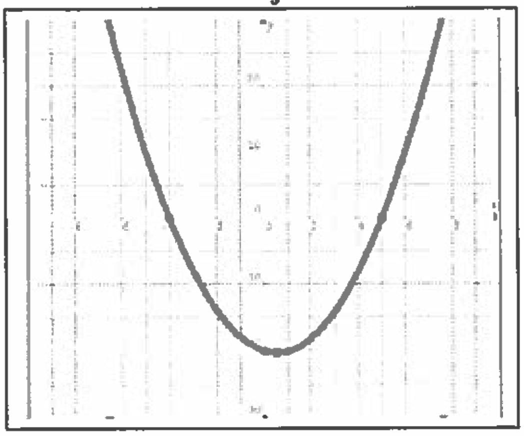
NOT a Function

Function notation ;

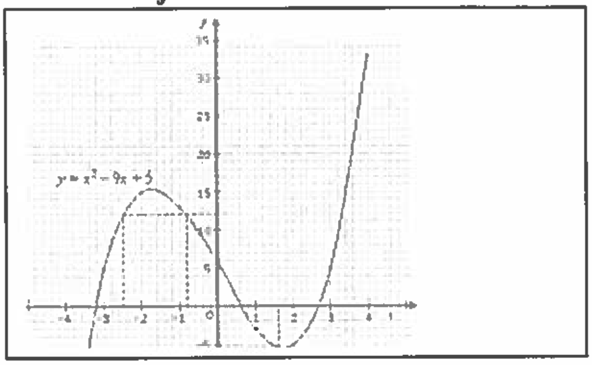
We are familiar with the notation $f(x) = x^2$ or $f : x \to f(x) : x \to x^2$.
Where in this case the rule is x goes in x^2 comes out.
For leaving cert we must define functions giving the Domain and Codomain as follows $f : R \to R : x \to f(x) = x^2$.
The first R is the **Domain** (the set of numbers that can be put into the function).
The second R is the **Codomain** (the set of numbers that can come out of the function).
The range of this function is R^+ (The set of numbers that actually come out of the function)
Knowledge of 3 types of function is required for Leaving Cert Higher Maths .
They are **(i)Surjective ,(ii) Injective (iii) Bijective** .
(i)Surjective : In this type of function the Range is equal to the Codomain .
That is there are no unused elements in the Codomain..
The function $f : R \to R : x \to f(x) = x^2$ is not surjective as all the elements in the **Codomain** are not occupied .If we were to change the Codomain to R^+ the function would be SurjectiveIn general if we draw a line of the form $y = k$ if the line does not intersect the graph of the function then that function is **not surjective**.

In general if we draw a line of the form $y = k$ the line must intersect the graph of the function at least once for the function to be **surjective**.

Not Surjective



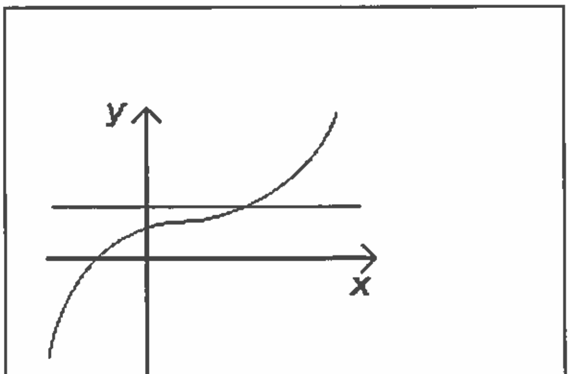
Surjective



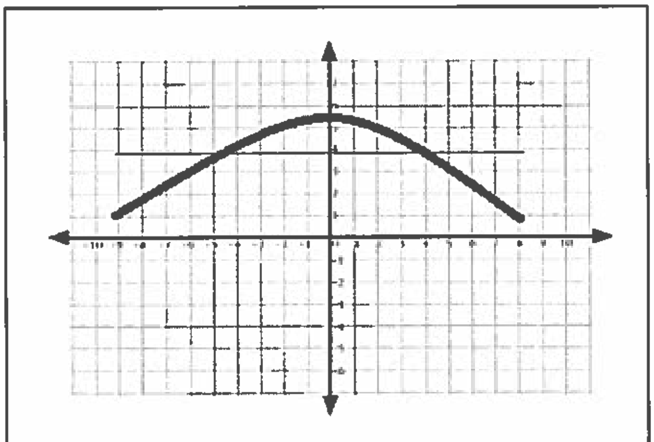
(ii)**Injective :**

A function is Injective if there is a 1to 1 mapping of each element of the Domain onto a Each element of the Codomain Note all the elements of the Codomain need not to be busy .

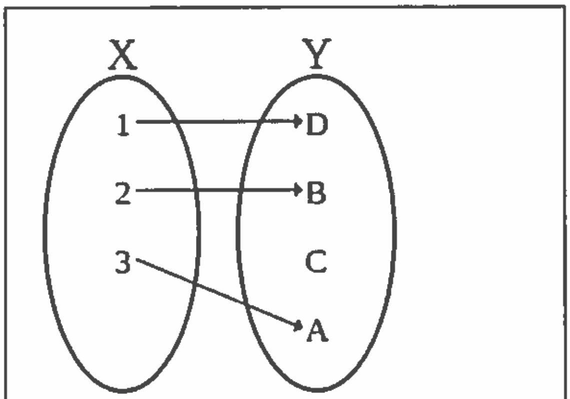
In general if we daw a line of the form $y = k$ it will cut the graph of $y = f(x)$ at most once.



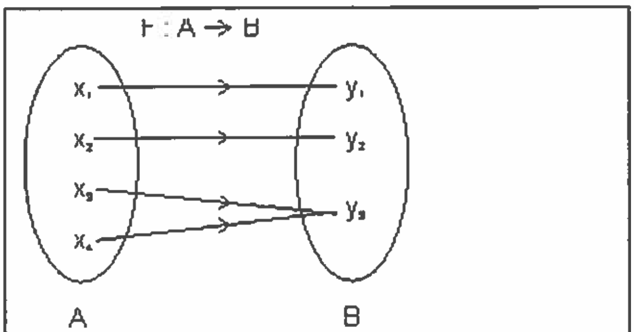
Injective



Non Injective



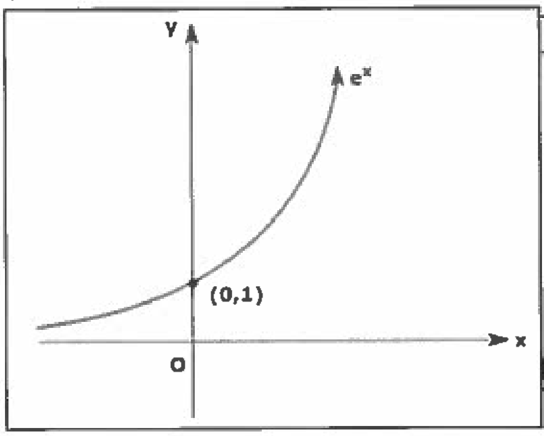
Injective

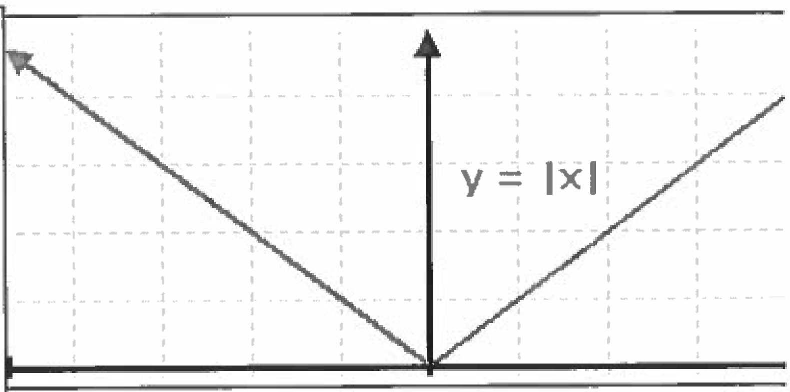


Non Injective

(iii)**Bijective ;**

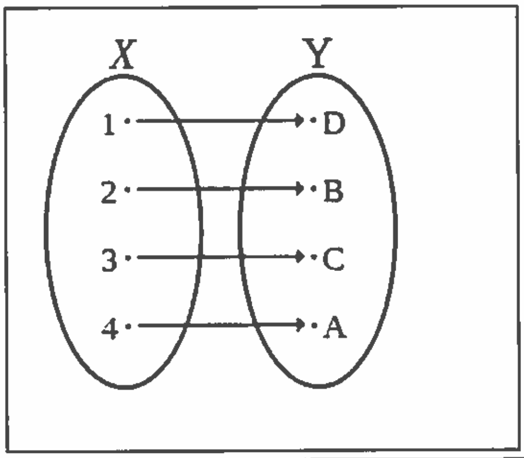
A function is said to be Bijective if it is both Surjective and Injective ,that is there is a 1to1 mapping from the Domain to the Codomain and the Range is equal to the Codomain (All the elements of the Codomain are busy).

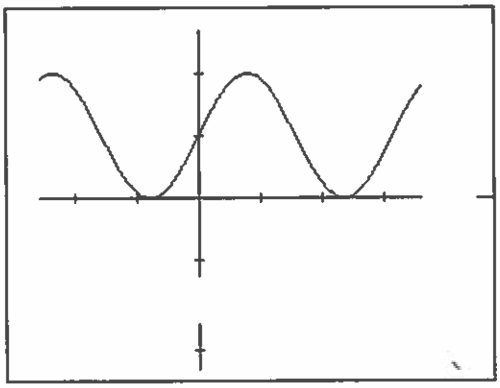


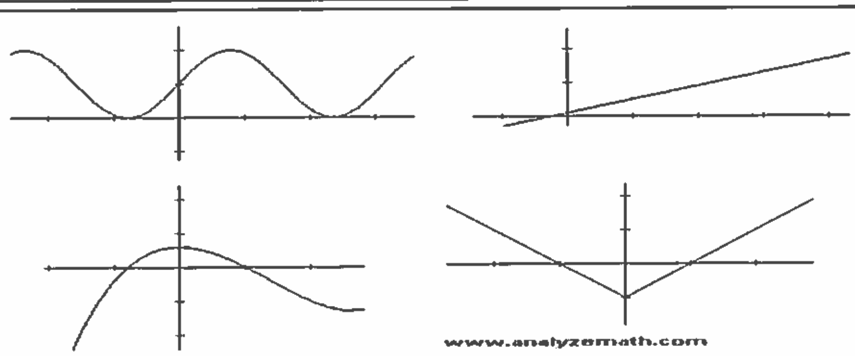


Function (x = k test) ,Surjection (y =k test)(,**Injective** (y = k test)
Therefore a Bijection

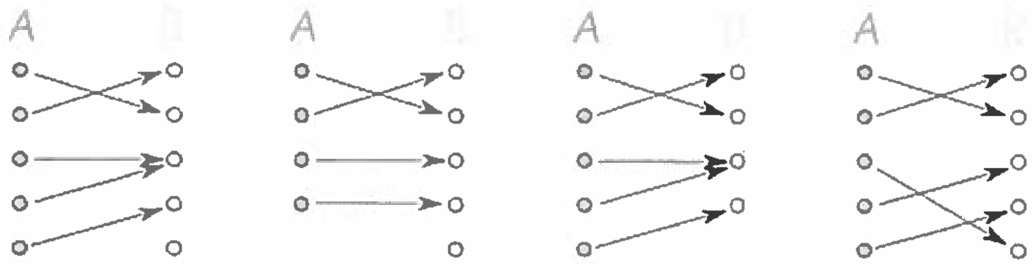
Function (x = k test) ,Surjection (y =k test)(not ,**Injective** (y = k test)
Therefore not a Bijection







(i)This function is not surjective or Injective
(ii)Function (ii)Is Surjective,Injective,and therefore Bijective
(iii)Function (iii) is not Surjective or ,Injective
Function (iv) is not Surjective , not Injective and therefore not Bijective.
Note the functions were defined as $f : R \to R : x \to f(x)$



General Function | Injective Not surjective | Surjective Not injective | Bijective (injective and surjective)

Composition of two functions

This is where one function is operated on the results of another function .
If f is a function such that $f(x)=3x+4$ and g is a function such that $g(x)=x^2$
The composite function is written as f after g this means we perform f after function g.
$f\circ g(x)=3x^2+4$.

Example (i) If $f(x)=3x+4$ and $g(x)=x^2$ find $f\circ g(5)$

$g(5)=(5)^2=25,..f(25)=3(25)+4=79\therefore f\circ g(5)=79$

Example (ii) If $f(x)=3x+4$ and $g(x)=x^2$ find $f\circ g(x)$

$g(x)=(x)^2,..f(x^2)=3(x^2)+4=3x^2+4\therefore f\circ g(x)=3x^2+4$.

Example (iii): Given $f(x)=2x^3+3,...g(x)=e^{3x}-2$. Find $f\circ g(0)$

$g(0)=e^{3(0)}-2=-1,..f(-1)=2(-1)^3+3=1$

Inverse functions

Only inverses of Bijective functions will be examined.

If f is the function $\{(a,1),(b,2),(c,3)\}$, $f^{-1}=\{(1,a),(2,b),(3,c)\}$ note f^{-1} is called f inverse and $f^{-1}\neq\frac{1}{f}$.Note that the domain of f is the range of f^{-1}.

Example (i): If

$f(x)=e^{2x}-5\rightarrow f^{-}:e^{2x}-5=x\rightarrow e^{2x}=x+5\rightarrow 2x\ln e=\ln(x+5)\rightarrow x=\frac{\ln(x+5)}{2}$

Example (ii) If $f(x)=2x^3+3$ find $f^{-1}(x)$

$f^{-1}:2x^3+3\rightarrow x\rightarrow 2x^3\rightarrow x-3\rightarrow x^3\rightarrow\frac{x-3}{2}\rightarrow x=\sqrt[3]{\frac{x-3}{2}}$

Differentiation and Integration of exponential functions.

Need to know the properties of e the exponential.

if $y = e^x \to \frac{dy}{dx} = e^x$. $y = \ln x \to \frac{dy}{dx} = \frac{1}{x}$

$(i).e^{\ln x} = x,..(ii).e^{\ln f(x)} = f(x)$, $a^x = e^{\ln a^x} = e^{x\ln a}$

To differentiate $y = a^x$ we have two methods at least.

Method (i) Using implicit differentiation (no longer on the course)

$$y = a^x \to \ln y = \ln a^x \to \ln y = x\ln a \to \frac{1}{y}\frac{dy}{dx} = \ln a \to \frac{dy}{dx} = y\ln a = a^x \ln a$$

Method (ii) $y = a^x \to y = e^{x\ln a} \to \frac{dy}{dx} = e^{x\ln a}\ln a = a^x \ln a$

To differentiate $y = \log_a x$ using the rules of logs for change of base

$$y = \log_a x = \frac{\log_e x}{\log_e a} \to \frac{dy}{dx} = \frac{1}{\log_e a}\frac{1}{x}$$ (note $\log_e a$ is a constant)

Integration of exponential functions.

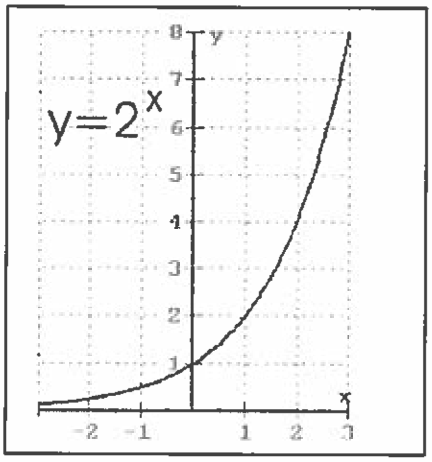
We know $\int e^x dx = e^x + c$ but how do we evaluate $\int a^x dx$?

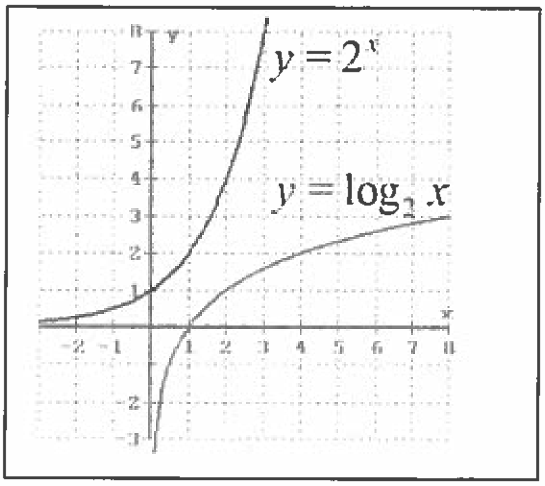
Method (i)Using the udu substitution (no longer on the course).

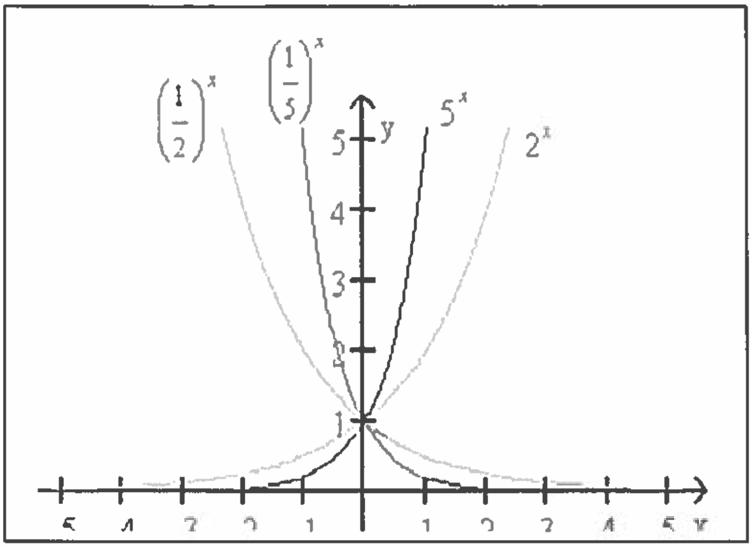
$$u = a^x \to \frac{du}{dx} = a^x \ln a \to \int a^x dx = \frac{1}{\ln a}\int du = \frac{1}{\ln a}(u) + c = \frac{1}{\ln a}a^x + c$$

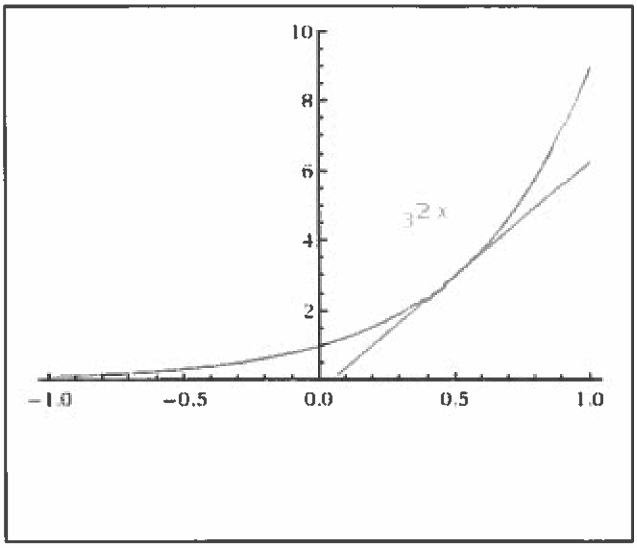
Method (ii) $\int a^x dx = \int e^{x\ln a} dx = \frac{1}{\ln a}e^{x\ln a} + c = \frac{1}{\ln a}a^x + c$

Graphs and properties of exponential functions $y = a^x, y = ba^x$







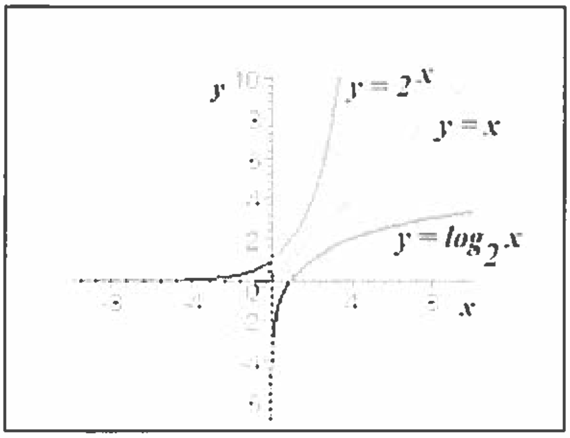


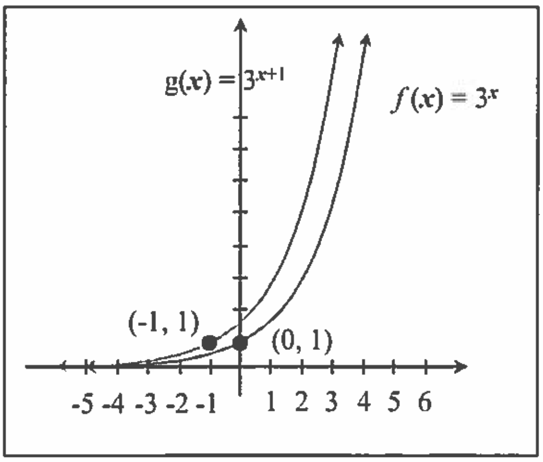
Properties of functions of the form $y = a^x, a > 1$

(i)The graph cuts the y axis at (01). (ii)As $x \to +\infty..a^x \to \infty$ (iii)As $x \to -\infty..a^x \to 0$
(iv)The X axis is the Horizontal Asymptote.
(v)Since $\frac{dy}{dx} = a^x \ln a$ then the slope of the tangent at any point $x = x_1$ is $f(x_1)\ln a$.(vi)The Inverse of $y = a^x$ is $y = \log_a x$
(v)The bigger the value of $a > 1$ the closer the graph is to the y axis.
(vi)If $0 < a < 1$ then as $x \to +\infty..a^x \to 0$ As $x \to -\infty..a^x \to \infty$
(vii) The domain of $f(x) = a^x$ consists of all real numbers.
(viii)The range of $f(x) = a^x$ consists of all positive real numbers
(ix)If $a > 1, f(x) = a^x$ has a graph that goes up to the right and is an increasing function.
(x) If $0 < a < 1, f(x) = a^x$ has a graph that goes down to the right and is a decreasing function.(xi) $f(x) = a^x$ is a one-to-one function and has an inverse that is a function. That is) $f(x) = a^x$ is a Bijection..

Properties of functions of the form $y = ba^x, a > 1, b > 0$

(i) The graph of this function cuts y axis at (0,b).
(ii) The slope of the tangent to the curve is $\frac{dy}{dx} = a^x b \ln a$, then the slope of the tangent at any point $x = x_1$ is $f(x_1) b \ln a$ (iii)The effect of the (b) is to stretch the graph vertically.



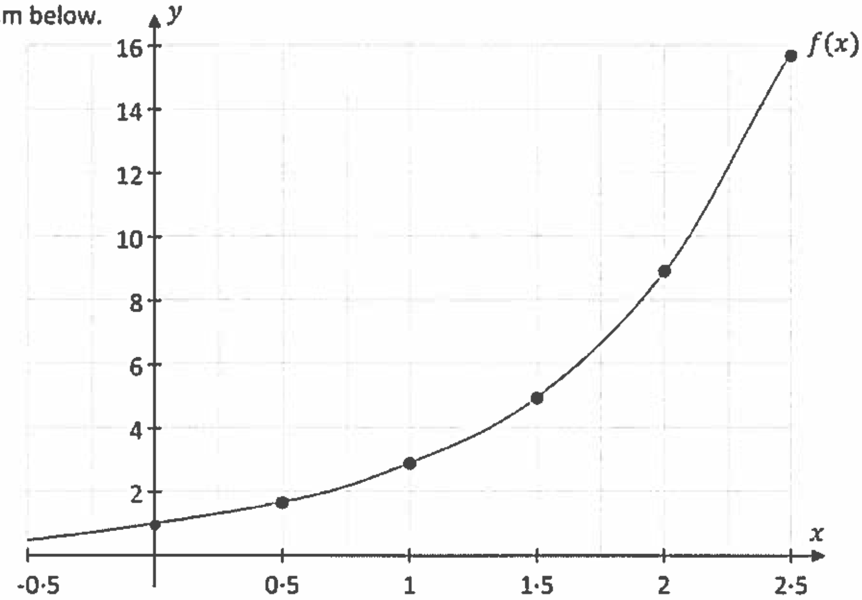


Functions and Calculus

2019 Question 2 – Higher Paper 1 Question

Question 2 **(25 marks)**

The graph of the function $f(x) = 3^x$, where $x \in \mathbb{R}$, cuts the y-axis at $(0, 1)$ as shown in the diagram below.



(a) **(i)** Draw the graph of the function $g(x) = 4x + 1$ on the diagram.

(ii) Use substitution to verify that $f(x) < g(x)$, for $x = 1{\cdot}9$.

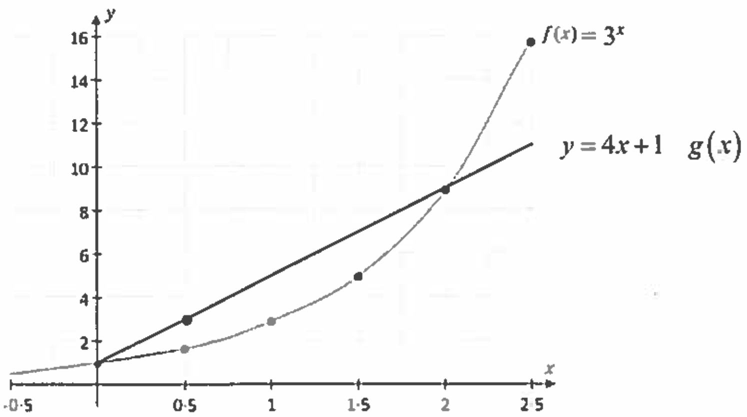
(b) Prove, using induction, that $f(n) \geq g(n)$, where $n \geq 2$ and $n \in \mathbb{N}$.

2019 Question 2 – Higher Paper 1 Solution

(a) (i) $g(x)=4x+1$ This is the line $y=4x+1$ Find 2 points on the line

$x=0.5, \quad y=4(0.5)+1 \quad \Rightarrow \quad y=3 \quad (0.5,3)$

$x=2, \quad y=4(2)+1 \quad \Rightarrow \quad y=9 \quad (2,9)$



(10 marks)

(ii) $f(x)<g(x) \quad \Rightarrow \quad 3^x<4x+1$

At $x=1.9$ just substitute $x=1.9$

$\Rightarrow \quad 3^{1.9}<4(1.9)+1 \quad \Rightarrow \quad 8.0636<8.6$ true **(5 marks)**

(b) To prove by induction the $f(n)\geq g(n)$

ie, $3^n\geq 4n+1$ for $n\geq 2$

Step 1: prove true for $n=2$: $3^2\geq 4(2)+1 \quad \Rightarrow \quad 9\geq 9$ true

Step 2: Assume true for $n=k$: $3^k\geq 4k+1$

Step 3: Prove true for $n=k+1$: $3^{k+1}\geq 4(k+1)+1 \quad \Rightarrow \quad 3\cdot 3^k>4k+4+1$

Hint: if the right-hand side is a sum, the left-hand side must also be a sum

Rewrite as $3^k+3^k+3^k\geq 4k+1+4$

$3^k\geq 4k+1$ from step 2 and $2\cdot 3k>4 \quad \Rightarrow \quad 3^k>2$ true

True for $n=2$ and true for $n=k+1$ $\therefore$ True for all n **(10 marks)**

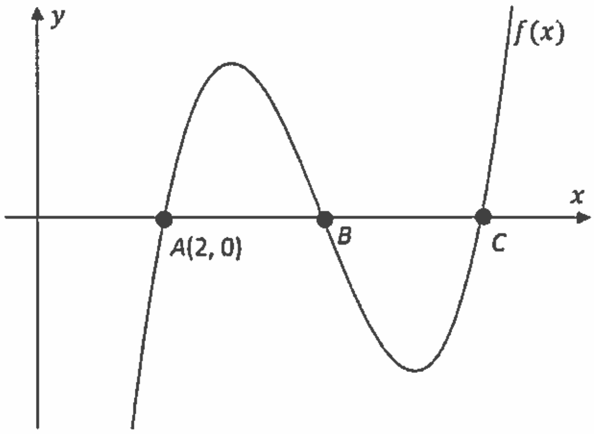
Comment: *The proof by induction may have caused problems.*

2019 Question 4 – Higher Paper 1 Question

Question 4 **(25 marks)**

(a) Find $\int (4x^3 - 6x + 10)\, dx$.

(b) Part of the graph of a cubic function $f(x)$ is shown below (graph not to scale).
The graph cuts the x-axis at the three points $A(2, 0)$, B, and C.



(i) Given that $f'(x) = 6x^2 - 54x + 109$, show that $f(x) = 2x^3 - 27x^2 + 109x - 126$.

(ii) Find the co-ordinates of the point B and the point C.

2019 Question 4 – Higher Paper 1 Solution

(a) $\int(4x^3-6x+10)dx=\frac{4x^4}{4}-\frac{6x^2}{2}+10x+c$

$\Rightarrow\ x^4-3x^2+10x+c$ **(10 marks)**

(b) (i) Given $f'(x)=6x^2-54x+109\ \Rightarrow\ f(x)=\int(6x^2-54x+109)dx$

$=\frac{6x^3}{3}-\frac{54x^2}{2}+109x+c=2x^3-27x^2+109x+c$

We know $A(2,0)$ is on the curve

$\therefore\ 2(2)^3-27(2)^2+109(2)+c=0$

$\Rightarrow\ 16-108+218+c=0\ \Rightarrow\ c=-126$

$\therefore\ f(x)=2x^3-27x^2+109x-126$ **(10 marks)**

(ii) To find B and C, we use the fact that $x=2$ is a root

$\therefore\ (x-2)$ is a factor. Divide by $(x-2)$

$$\begin{array}{r l}
 & 2x^2-23x+63 \\
x-2\,) & 2x^3-27x^2+109x-126 \\
 & \underline{2x^3-4x^2} \quad \text{subtract} \\
 & -23x^2+109x \\
 & \underline{-23x^2+46x} \quad \text{subtract} \\
 & 63x-126 \\
 & \underline{63x-126} \quad \text{subtract} \\
 & 0 \quad 0
\end{array}$$

$\therefore$ The second factor is $2x^2-23x+63$

Now solve $2x^2-23x+63=0\ \Rightarrow\ (2x-9)(x-7)=0$

$x=4{,}5,\quad x=7\quad B$ is (4.5,0), C is (7,0)

You could also have used $x=\frac{-b\mp\sqrt{b^2-4ac}}{2a}$ **(5 marks)**

Comment: *Not a bad question.*

2019 Question 8 – Higher Paper 1 Question

Question 8 **(50 marks)**

The weekly revenue produced by a company manufacturing air conditioning units is seasonal. The revenue (in euro) can be approximated by the function:

$$r(t) = 22\,500 \cos\left(\frac{\pi}{26}t\right) + 37\,500, \qquad t \geq 0$$

where t is the number of weeks measured from the beginning of July and $\left(\frac{\pi}{26}t\right)$ is in radians.

(a) Find the approximate revenue produced 20 weeks after the beginning of July. Give your answer correct to the nearest euro.

(b) Find the two values of the time t, within the first 52 weeks, when the revenue is approximately €26 250.

(c) Find $r'(t)$, the derivative of $r(t) = 22\,500 \cos\left(\frac{\pi}{26}t\right) + 37\,500$.

(d) Use calculus to show that the revenue is increasing 30 weeks after the beginning of July.

(e) Find a value for the time t, within the first 52 weeks, when the revenue is at a minimum. Use $r''(t)$, to verify your answer.

2019 Question 8 – Higher Paper 1 Solution

(a) $r(t) = 22{,}500\cos\left(\frac{\pi}{26}t\right) + 37{,}500$

(note this is a periodic function in the form $a + b\cos kt$)

t is the number of weeks measures from the beginning of July

$r(26) = 22{,}500\cos\left(\frac{\pi}{26}(26)\right) + 37{,}500 = €20{,}659$ **(10 marks)**

(b) Find t if revenue is €26,250

$\Rightarrow\ 26{,}250 = 22{,}500\left(\cos\frac{\pi}{26}t\right) + 37{,}500 \quad \Rightarrow\ 26{,}250 - 37{,}500 = 22{,}500\left(\cos\frac{\pi}{26}t\right)$

$\Rightarrow\ \frac{26{,}250 - 37{,}500}{22{,}500} = \cos\frac{\pi}{26}t = -\frac{1}{2} \quad \Rightarrow\ \frac{\pi}{26}t = \cos^{-1}\left(-\frac{1}{2}\right)$

$\frac{\pi}{26}t = \frac{2\pi}{3} \quad \Rightarrow\ t = \frac{2(26)}{3} = \frac{52}{3} = 17\frac{1}{3}$ and $\frac{\pi}{26}t = \frac{4\pi}{3} \quad \Rightarrow\ t = \frac{104}{3} = 34.6$

(10 marks)

(c) $r(t) = 22{,}500\cos\left(\frac{\pi}{26}t\right) + 37{,}500$

$r'(t) = 22{,}500\left(-\sin\frac{\pi}{26}t\right)\left(\frac{\pi}{26}\right)$ **(10 marks)**

(d) If $r'(30) > 0 \quad \Rightarrow\quad r$ is increasing

$r'(30) = 22{,}500\left(-\sin\frac{\pi}{26}(30)\right)\frac{\pi}{26} = 1263.43 > 0$

$\therefore$ The revenue is increasing 30 weeks after the beginning of July **(10 marks)**

(e) To find the minimum, solve $r'(t) = 0 \quad \Rightarrow\quad 22{,}500\left(-\sin\frac{\pi}{26}t\right)\frac{\pi}{26} = 0$

$\Rightarrow\ -\sin\frac{\pi}{26}t = 0 \quad \Rightarrow\ \frac{\pi}{26}t = \pi \quad \Rightarrow\ t = 26$

Or, we know that the range of $22{,}500\cos\left(\frac{\pi}{26}t\right) + 37{,}500$ is

37,500 – 22,500 to 37,500 + 22,500

(Minimum) 15,000 to 60,000 (maximum)

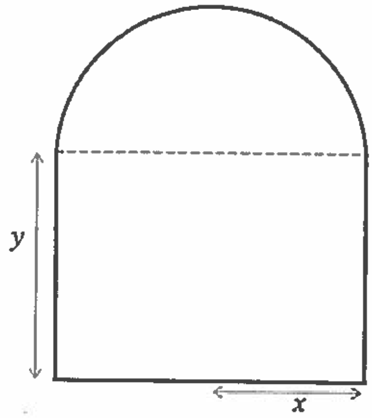
The minimum occurs when $\cos\frac{\pi}{26}t = -1 \quad \Rightarrow\quad t = 26$, i.e. at $t = 26$

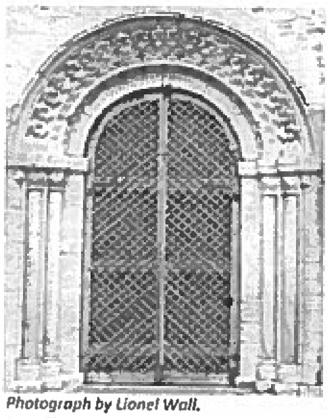
$r''(26) > 0 \quad \Rightarrow\quad r$ is minimum **(10 marks)**

Comment: *Difficult question. Very messy.*

2019 Question 9 – Higher Paper 1 Question

Question 9 **(55 marks)**





Photograph by Lionel Wall.
http://greatenglishchurches.co.uk/html/castle_rising/html

Norman windows consist of a rectangle topped by a semi-circle as shown above.
Let the height **of the rectangle** be y metres and the radius of the semi-circle be x metres as shown. The perimeter of the window is P.

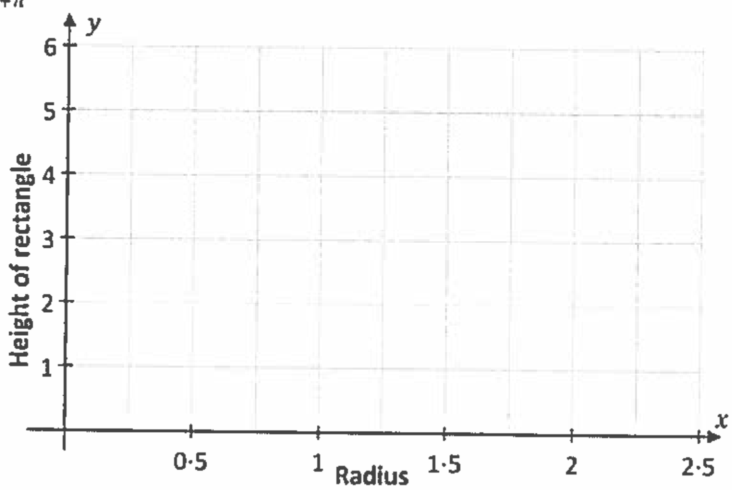
(a) (i) Write P in terms of x, y, and π.

(ii) In a particular Norman window the perimeter $P = 12$ metres.
Show that $y = \frac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.

(b) (i) Complete the table on the right.

x	0	$\frac{12}{2+\pi}$
$y = \frac{12-(2+\pi)x}{2}$		

(ii) On the diagram below, draw the graph of the linear function, $y = \frac{12-(2+\pi)x}{2}$ for $0 \leq x \leq \frac{12}{2+\pi}$ where $x \in \mathbb{R}$.



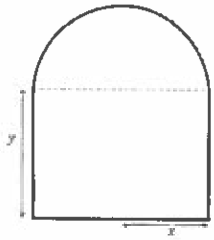
(iii) Find the slope of the graph of y, correct to 2 decimal places.
Interpret this slope in the context of the question.

2019 Question 9 – Higher Paper 1 Solution

(a) (i) The perimeter is $y+2x+y+\pi x$

$P=2y+2x+\pi x$

$P=2y+x(2+\pi)$



(10 marks)

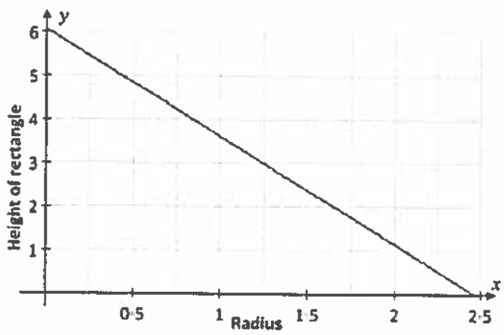
(ii) Given $P=12 \Rightarrow 12=2y+x(2+\pi) \Rightarrow \dfrac{12-x(2+\pi)}{2}=y$ **(10 marks)**

(b) (i)

x	0	$\dfrac{12}{2+\pi}$
$y=\dfrac{12-(2+\pi)x}{2}$	6	0

(5 marks)

(ii)



(5 marks)

(iii) To find the slope of the graph. This is a linear equation of the form $y=mx+c$

$y=6-\dfrac{x(2+\pi)}{2}$

$\therefore$ The slope is $\dfrac{-2-\pi}{2}=-2.57$

Or $\dfrac{y_2-y_1}{x_2-x_1}=\dfrac{6-0}{0-\dfrac{12}{2+\pi}}=\dfrac{-2-\pi}{6}=-2.57$

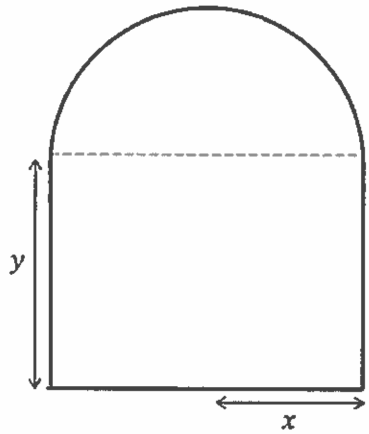
Interpretation: function is decreasing **(10 marks)**

2019 Question 9 – Higher Paper 1 Question (*continued*)

(c) (i) The Norman window shown below has a perimeter of 12 metres and $y = \frac{12-(2+\pi)x}{2}$.

Show that the function $a(x) = \frac{24x-(\pi+4)x^2}{2}$ represents the area of the window, in terms of x and π.

(ii) Find $a'(x)$.



(iii) Find the relationship between x and y when the area of the window in **part (c)(i)** is at its maximum.

2019 Question 9 – Higher Paper 1 Solution (*continued*)

(c) (i) The area of the window is the area of the rectangle plus the area of the semi-circle:

Area $= 2xy + \frac{\pi x^2}{2}$ We know $y = \frac{12-(2+\pi)x}{2}$

$A = \left(\frac{12-2x-x\pi}{2}\right)(2x) + \frac{\pi x^2}{2}$

$= \frac{24x - 4x^2 - 2x^2\pi + \pi x^2}{2}$

$\frac{24x - 4x^2 - \pi x^2}{2} = \frac{24x - x^2(4+\pi)}{2}$ **(10 marks)**

(ii) Set $a' = 0$

$a' = \left(\frac{da}{dx}\right) = \frac{1}{2}(24 - 2x(4+\pi)) = 0 \Rightarrow 12 - x(4+\pi) = 0$

$\Rightarrow x = \frac{12}{4+\pi}$

Now find y

$y = \frac{12-(2+\pi)x}{2} \Rightarrow y = \frac{12-(2+\pi)\frac{12}{\pi+4}}{2} = \frac{12}{\pi+4}$

$x = \frac{12}{\pi+4},\ y = \frac{12}{\pi+4} \Rightarrow |x| = |y|$ **(5 marks)**

Comment: *Nice question.*

2018 Question 3 – Higher Paper 1 Question

Question 3 **(25 marks)**

(a) Let $h(x) = \cos(2x)$, where $x \in \mathbb{R}$.
A tangent is drawn to the graph of $h(x)$ at the point where $x = \frac{\pi}{3}$.
Find the angle that this tangent makes with the positive sense of the x-axis.

(b) Find the average value of $h(x)$ over the interval $0 \leq x \leq \frac{\pi}{4}$, $x \in \mathbb{R}$.
Give your answer in terms of π.

2018 Question 3 – Higher Paper 1 Solution

(a) Given $h(x) = \cos 2x$

$\frac{dy}{dx}$ gives the slope of the tangent

$h'(x) = -2\sin 2x$ When $x = \frac{\pi}{3}$, $\frac{dy}{dx} = -2\sin 2\left(\frac{\pi}{3}\right) = -2\sin\left(\frac{2\pi}{3}\right) = -2\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$

$\therefore \tan\theta = -\sqrt{3}$

where θ is the angle the tangent makes with the positive sense of the x axis

$\tan\theta = -\sqrt{3} \quad \Rightarrow \quad \theta = 120^\circ$ **(15 marks)**

(b) The average value of $f(x)$ over the interval $a \le x \le b$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{(\pi/4)-0}\int_0^{\pi/4} \cos 2x\,dx = \frac{1}{(\pi/4)-0}\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4}$$

$\frac{4}{\pi} \times \frac{1}{2}\left[\sin 2(\pi/4) - \sin 2(0)\right]$ Tidy up

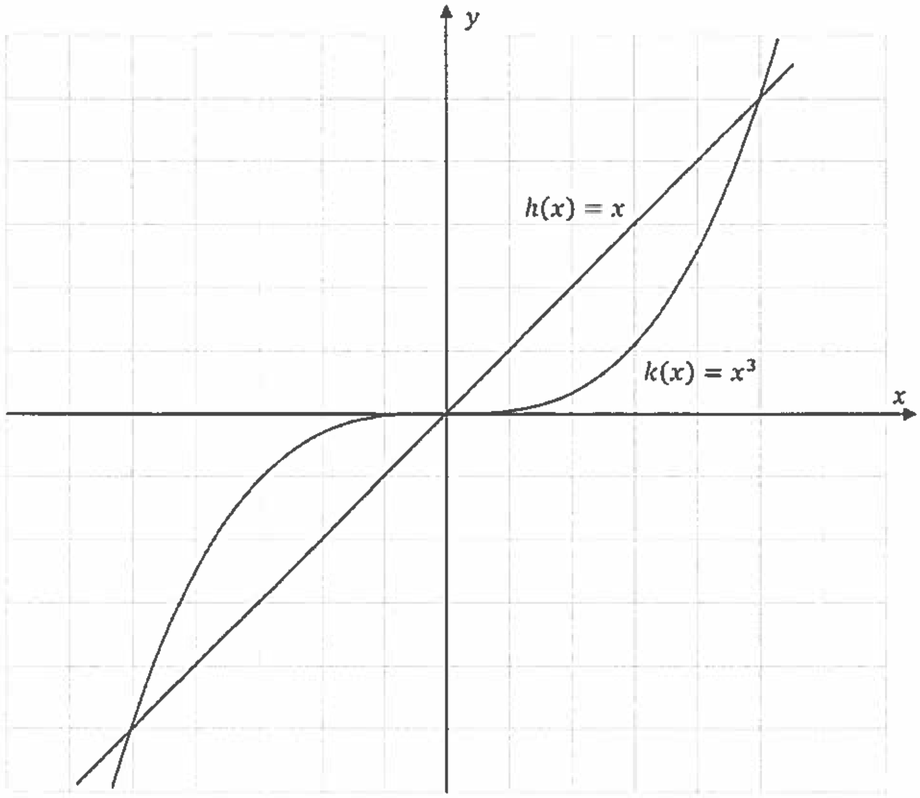
$\frac{2}{\pi}[1] = \frac{2}{\pi}$ **(10 marks)**

Comment: *Very nice question. Short and concise. Just make sure your calculator is in radian mode.*

2018 Question 6 – Higher Paper 1 Question

Question 6 **(25 marks)**

Parts of the graphs of the functions $h(x) = x$ and $k(x) = x^3$, $x \in \mathbb{R}$, are shown in the diagram below.



(a) Find the co-ordinates of the points of intersection of the graphs of the two functions.

(b) (i) Find the total area enclosed between the graphs of the two functions.

(ii) On the diagram on the previous page, using symmetry or otherwise, draw the graph of k^{-1}, the inverse function of k.

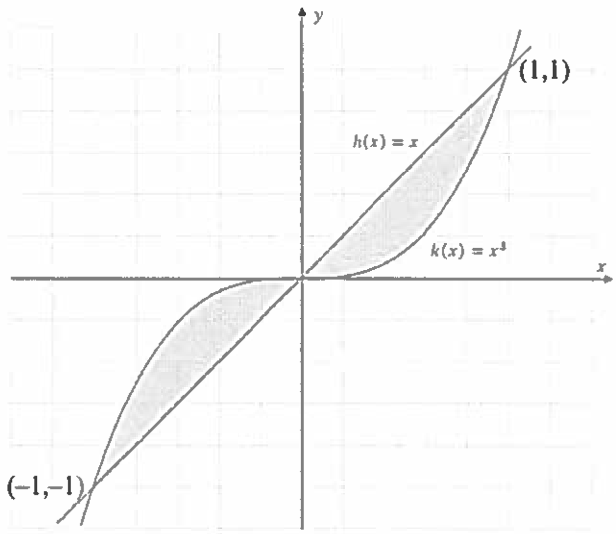
2018 Question 6 – Higher Paper 1 Solution

(a) To find the points of intersection of $h(x)$ and $k(x)$, solve $x(x-1)(x+1)=0$

$k(x)=h(x)=x^3-x=x(x^2-1)=0$

$\therefore\ x=0,\quad x=1,\quad x=-1\quad (0,0),\ (1,1),\ (-1,-1)$ **(10 marks)**

(b) To find the area between the Graphs, you can use the fact that because the graphs are symmetrical, we can just find the area under $h(x)$ – the area under $k(x)$



$$2\int_0^1 x-x^3dx=2\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

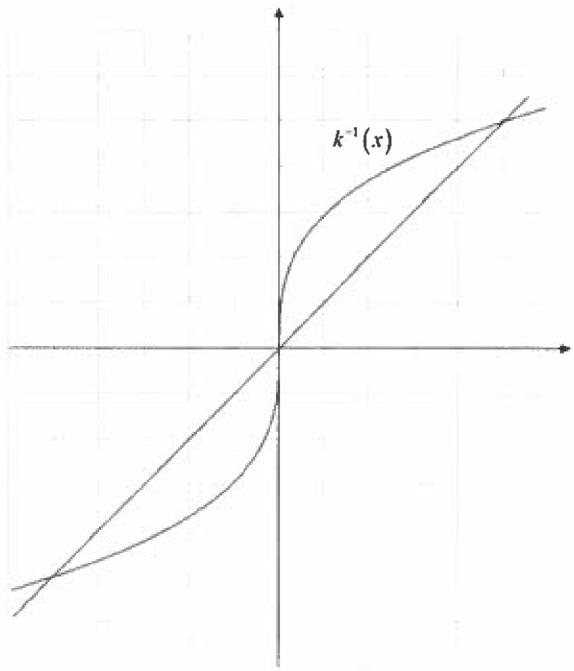
$$2\left[\frac{1}{2}-\frac{1}{4}\right]-\left[\frac{0^2}{2}-\frac{0^4}{4}\right]=\frac{1}{2}$$

$k(x)=x^3$ i.e. $y=x^3$

To find $k^{-1}(x)$, swop x and y

$x=y^3 \quad\Rightarrow\quad y=\sqrt[3]{x}$ To find this graph, reflect $k(x)$ in the line $y=x(h(x))$

to get the graph of $k^{-1}(x)$



(15 marks)

Comment: *Some students just read the points of intersection from the graph and got (–5,–5) and (5,5). Nice question. Most should be happy.*

2017 Question 5 – Higher Paper 1 Question

Question 5 **(25 marks)**

The function f is such that $f(x) = 2x^3 + 5x^2 - 4x - 3$, where $x \in \mathbb{R}$.

(a) Show that $x = -3$ is a root of $f(x)$ and find the other two roots.

(b) Find the co-ordinates of the local maximum point and the local minimum point of the function f.

(c) $f(x) + a$, where a is a constant, has only one real root.
Find the range of possible values of a.

2017 Question 5 – Higher Paper 1 Solution

(a) $f(x)=2x^3+5x^2-4x-3$ To show $x=-3$ is a root, show $f(-3)=0$

$f(-3)=2(-3)^3+5(-3)^2-4(-3)-3=-54+45+12-3=0$

$\therefore\ x=-3$ is a root $\Rightarrow$ $x+3$ is a factor

Now divide $f(x)$ by $(x+3)$ to find the other roots

$$\begin{array}{r|l} & 2x^2-x-1 \\ x+3 & 2x^3+5x^2-4x-3 \\ & \underline{2x^3+6x^2} \\ & \quad -x^2-4x \\ & \quad \underline{-x^2-3x} \\ & \qquad -x-3 \\ & \qquad \underline{-x-3} \\ & \qquad 0\quad 0 \end{array}$$

Now solve $2x^2-x-1=0$ $\Rightarrow$ $(2x+1)(x-1)=0$

$x=-\frac{1}{2},\quad x=1\ \therefore$ 3 roots are $x=-3,\quad -\frac{1}{2},\quad 1$ **(15 marks)**

(b) $f'(x)=6x^2+10x-4=0$ $\Rightarrow$ $3x^2+5x-2=0$

$(3x-1)(x+2)=0$ $\Rightarrow$ $x=\frac{1}{3},\quad x=-2$

$f''(x)=12x+10\qquad f''\left(\frac{1}{3}\right)=12\left(\frac{1}{3}\right)+10=14$ min as $f''(x)>0$

$f''(-2)=12(-2)+10=-14$ max as $f''(x)<0$

Now find y: $f\left(\frac{1}{3}\right)=\frac{-100}{27}$ $\therefore$ Min at $\left(\frac{1}{3},\frac{-100}{27}\right)$

Max $f(-2)=9$ $\therefore$ Max at $(-2,9)$ **(5 marks)**

(c) New max and min of $f(x)+a$

$f(x)+a$ will have a max at $(-2,9+a)$ and a min at $\left(\frac{1}{3},\frac{-100}{27}+a\right)$

One of the conditions that a cubic equation has only one real root is

$(y_{\max})(y_{\min})>0$ $\Rightarrow$ $(9+a)\left(\frac{-100}{27}+a\right)>0$ $\Rightarrow$ $-9>a$ or $a>\frac{100}{27}$

Another condition is $f'(x)$ must be >0 (function is increasing) **(5 marks)**

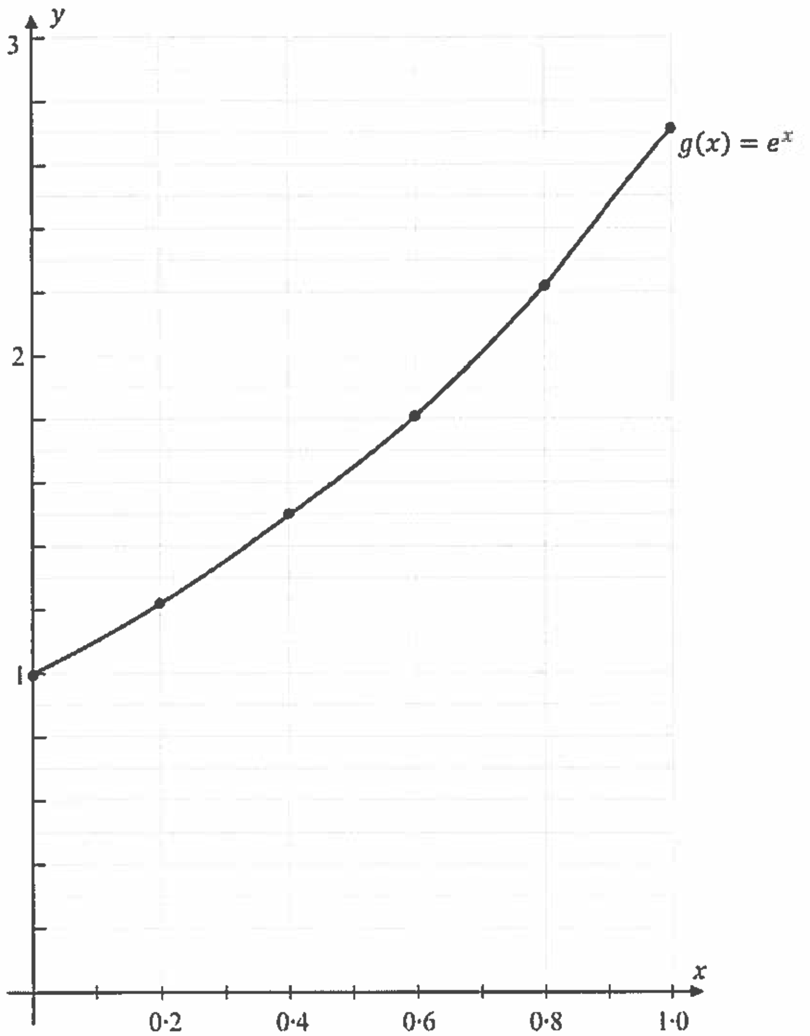
Comment: *Part (c) would have caused problems.*

2017 Question 6 – Higher Paper 1 Question

Question 6 **(25 marks)**

The graph of the function $g(x) = e^x, x \in \mathbb{R}, 0 \leq x \leq 1$, is shown on the diagram below.

(a) On the same diagram, draw the graph of $h(x) = e^{-x}, x \in \mathbb{R}$, in the domain $0 \leq x \leq 1$.



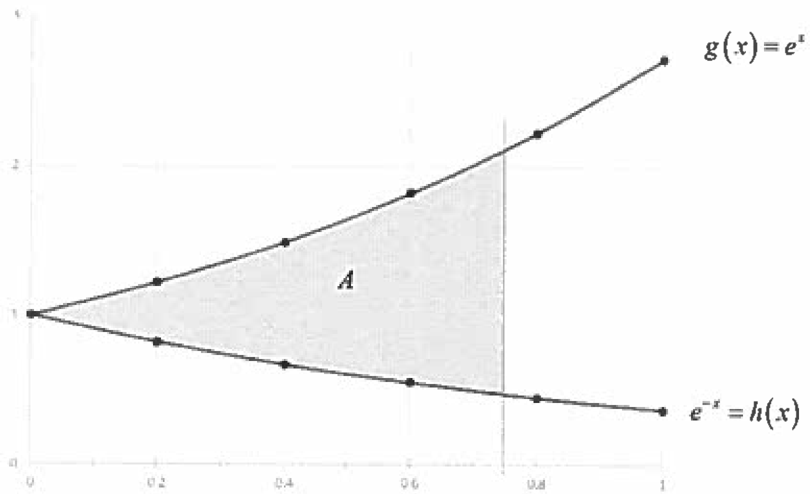
(b) Find the area enclosed by $g(x) = e^x$, $h(x) = e^{-x}$, and the line $x = 0{\cdot}75$.
Give your answer correct to 4 decimal places.

2017 Question 6 – Higher Paper 1 Solution

(a) To sketch the graph of $y = e^{-x}$, use the table funcion on your calculator

x	0	0.2	0.4	0.6	0.8	1
e^{-x}	1	0.82	0.67	0.55	0.45	0.37

(10 marks)



A is the area below $g(x)$ – area below $h(x)$ between $x = 0$ and $x = 0.75$

$A = \int_0^{0.75} e^x - e^{-x} dx = \left[e^x + e^{-x}\right]_0^{0.75}$

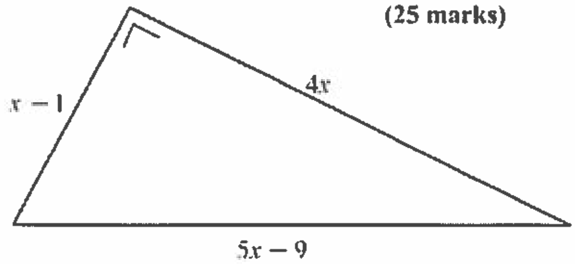
$e^{0.75} + e^{-0.75} - \left[e^0 + e^0\right] = 0.5894$ **(15 marks)**

Comment: *Nice question. Very short. The table key on the Casio is a great asset.*

2016 Question 5 – Higher Paper 1 Question

Question 5 **(25 marks)**

(a) (i) The lengths of the sides of a right-angled triangle are given by the expressions $x-1$, $4x$, and $5x-9$, as shown in the diagram. Find the value of x.



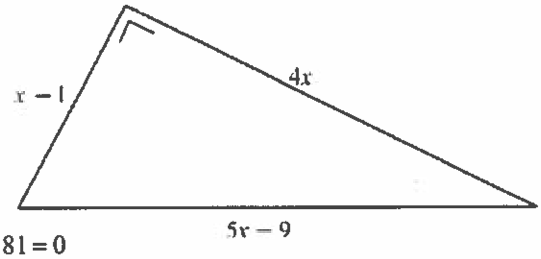
(ii) Verify, with this value of x, that the lengths of the sides of the triangle above form a pythagorean triple.

(b) (i) Show that $f(x) = 3x - 2$, where $x \in \mathbb{R}$, is an injective function.

(ii) Given that $f(x) = 3x - 2$, where $x \in \mathbb{R}$, find a formula for f^{-1}, the inverse function of f. Show your work.

2016 Question 5 – Higher Paper 1 Solution

(a) (i) Use Pythagoras to find x



$(x-1)^2+(4x)^2=(5x-9)^2$

$x^2-2x+1+16x^2=25x^2-90x+81$

$\Rightarrow\ x^2-2x+1+16x^2-25x^2+90x-81=0$

$\Rightarrow\ -8x^2+88x-80=0\ \Rightarrow\ x^2-11x+10=0$

$\Rightarrow\ (x-10)(x-1)=0\ \Rightarrow\ x=10,\quad x=1$

Note $x=1$ is inadmissable as it makes $x-1=0$. Be careful here.

This issue was mentioned in the 2015 Chief Examiners report. Always test your answer.

$\therefore$ Solution is $x = 10$ **(10 marks)**

(ii) This gives sides 9, 40, 41

$41^2=40^2+9^2\ \Rightarrow\ 1681=1681$ **(5 marks)**

(b) (i) To show $f(x)=3x-2$ is injective, sketch $y=3x-2$. Just use 2 points

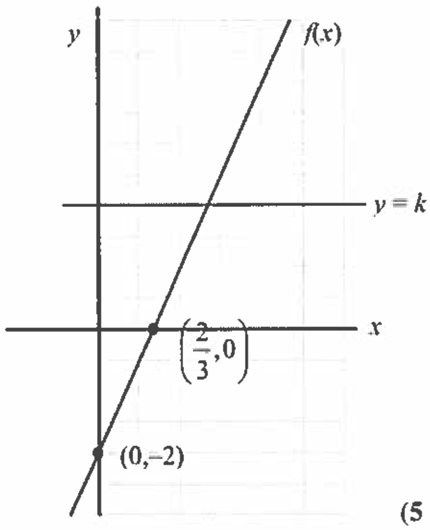
$x=0,\quad y=-2$ and $y=0,\quad x=\frac{2}{3}$

The horizontal line $y=k$

will cut the graph of $f(x)$ at most once

$\Rightarrow\ f(x)$ is injective

(injective means $1\rightarrow1$)



(5 marks)

(ii) $f(x)=3x-2$ To find the inverse, write as $y=3x-2$

Now swap x and y to get $x=3y-2$

Isolate $y\ \Rightarrow\ \frac{x+2}{3}=y$

$\therefore\ f^{-1}(x)=\frac{x+2}{3}$ **(5 marks)**

Comment: *Very short question.*